CHNOLOGY DEPARTMENT

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 66 NUMBER 3

CONTENTS

Recent Results in Surface Area Theory . . . LAMBERTO CESARI 173
On the Solution of the Differential Equation $f(x, y, y') = 0$ . . . . . SMBAT ABIAN AND ARTHUR B. BROWN 192
Prouhet's 1851 Solution of the Tarry-Escott Problem of 1910. . . . . E. M. WRIGHT 199
A Look at Mathematical Competitions . . R. CREIGHTON BUCK 201
Conference on the Committee on the Undergraduate Program . . 213
Mathematical Notes . . . . .S. K. CHATTERJEA, J. D. WESTON 220
Classroom Notes. . . . . J. C. SALTZER, GALLEGO-DIAZ 225
Mathematical Education Notes C. F. BRUMFIEL, CHARLES C. BUCK 228
Elementary Problems and Solutions 233
Advanced Problems and Solutions 238
Recent Publications 243
News and Notices 249
The Mathematical Association of America 252
New Members. 252
October Meeting of the Iowa Section 254
October Meeting of the Minnesota Section 255
October Meeting of the Oklahoma Section 256
November Meeting of the Northeastern Section 257
Calendar of Future Meetings 258

MARCH 1959

PUBLIC LIBRARY
APR 8 1959
DETROIT

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

RALPH D. JAMES, *Editor*

ASSOCIATE EDITORS

RICHARD V. ANDREE	G. E. HAY	C. O. OAKLEY
JOHN A. BROWN	L. M. KELLY	J. M. H. OLMSTED
ROY DUBISCH	JOHN R. MAYOR	E. P. STARKE
HOWARD EVES	LLOYD J. MONTZINGO, JR.	H. S. ZUCKERMAN
	IVAN NIVEN	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. JAMES, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.
(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, C. B. ALLENDOERFER, University of Washington
First Vice-President, G. B. THOMAS, JR., Massachusetts Institute of Technology
Second Vice-President, HARLEY FLANDERS, University of California
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, LLOYD J. MONTZINGO, JR., University of Buffalo
Editor, R. D. JAMES, University of British Columbia

Additional Members of the Board of Governors: H. M. BACON, JEWELL H. BUSHEY, LAMBERTO CESARI, R. V. CHURCHILL, H. S. M. COXETER, P. H. DAUS, W. L. DUREN, JR., H. P. EVANS, HOWARD EVES, WILLIAM FELLER, N. J. FINE, J. S. FRAME, G. B. HUFF, W. N. HUFF, G. K. KALISCH, E. C. KIEFER, MORRIS KLINE, J. C. KNIPP, W. G. LEAVITT, Z. L. LOFLIN, A. T. LONSETH, J. R. MAYOR, E. J. MCSHANE, G. M. MERRIMAN, R. S. PARK, GEORGE PÓLYA, G. B. PRICE, O. J. RAMLER, C. R. SHERER, R. G. SMITH, B. M. STEWART, F. M. STEWART, W. R. UTZ, JR., BERNARD VINOGRADE, CHARLES WEXLER, C. R. WYLIE, JR.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For non-members the subscription price is $6.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y.
during the months of January, February, March, April, May, June-July,
August-September, October, November, December.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.
Second-class postage paid at Menasha, Wisconsin.

Copyright © 1959 by the Mathematical Association of America, Inc.

PRINTED IN THE UNITED STATES OF AMERICA

RECENT RESULTS IN SURFACE AREA THEORY

LAMBERTO CESARI, Purdue University

There is a growing interest in a general theory concerning the analytical properties of transformations, or mappings

$$(1) \qquad (T, A): p = p(w),\ w \in A \subset X,\ p \in Y, \text{ or } T: A \to Y,$$

from a set A of a "space X" into a "space Y." Let us say explicitly that T is meant to be single valued but not necessarily one-one; that is, each $w \in A$ is mapped into one and only one $p = p(w) \in Y$ (image of w), that all these $p = p(w)$, $w \in A$, form a set $T(A) \subset Y$ (graph) of T, but each $p \in T(A)$ may be the image of more than one $w \in A$, even infinitely many $w \in A$ (counter images of p). Real functions of one real variable, parametric curves, surfaces, *etc.*, are examples of such mappings, and the analytical entities attached to (T, A) may be called total variation, length, area, *etc.*, or, more generally, line integrals, surface integrals, *etc.*

The last concepts are usually introduced under very restrictive conditions on T, but recently it has been recognized that length, area, *etc.*, can be introduced under the mere hypothesis of continuity of T (and even this may not be required), and that the finiteness of the length [area, *etc.*] assures the existence of a line integral [surface integral, *etc.*].

This viewpoint has been illustrated for curves in a previous article [II].* Here we discuss the analogous approach for surfaces, which entails a much deeper connection with topology and measure theory than for curves. Total variation of real functions of one real variable and related concepts have been illustrated in another article [I].†

1. The concept of surfaces. We shall denote by E_2 the real Euclidean w-plane, $w = (u, v)$, by E_N any real Euclidean space (for $N = 3$ let E_3 be the p-space with $p = (x, y, z)$), by $\overline{M}$, M^*, M^0 the closure, the boundary, the set of the interior points of a set M inone such space, by $|p| = (x^2+y^2+z^2)^{1/2}$ the Euclidean norm, and by $|p-q|$ the Euclidean distance of two points p, q.

By a *surface* S we shall mean a mapping (1) from some set $A \subset E_2$ into E_3 (or E_N), where A could be, for instance, a square, a circle, a polygonal region, or a (closed) simple Jordan region J, or more generally, a (closed) Jordan region of finite connectivity $\nu \geqq 0$, say $J = J_0 - (J_1 + \cdots + J_\nu)^0$, $J_i \subset J_0^0$, $J_i J_j = 0$, $i \neq j$, $i = 1, \cdots, \nu$, where all J_i and J_0 are closed simple Jordan regions. It has been found convenient to take for A any "admissible" set, *i.e.*, either any Jordan region J as above, or a finite sum of disjoint Jordan regions, or any set $G \subset E_2$, open in E_2, or any set $G \subset J$, open in J (further generalizations have been and

* Lamberto Cesari, Rectifiable curves and the Weierstrass integral, this MONTHLY, vol. 65, 1958, pp. 485–500. This article will be quoted as [II].

† Lamberto Cesari, Variation, multiplicity, and semicontinuity, this MONTHLY, vol. 65, 1958, pp. 317–332. This article will be quoted as [I].

are being studied). We will suppose $N=3$. Thus (T, A) is defined by a continuous vector function $T(w)$, $w\in A$, say

$$(1.1)\qquad \begin{aligned} &S=(T, A): p=T(w),\ w\in A,\ T(w)=[x(w), y(w), x(w)], \text{ or}\\ &S=(T, A): x=x(u, v),\ y=y(u, v),\ z=z(u, v),\ (u, v)\in A. \end{aligned}$$

The set $T(A)\subset E_3$ is the graph of $S=(T, A)$, but in no way defines (T, A). For instance the mappings $T: x=u,\ y=v,\ z=0,\ (u, v)\in Q$, and $T': x=\sin \frac{1}{2}k\pi u,\ y=v, z=0,\ (u, v)\in Q$, where Q is the square $Q=[0\leqq u, v\leqq 1]$ and $k>1$ an integer, have both the same graph $T(Q)=T'(Q)=U$, the unit square $U=[0\leqq x,\ y\leqq 1,\ z=0]$ in the xy-plane, but T covers U just once, while T' covers U exactly k times. Finally, if $P: x=\phi(t),\ y=\psi(t),\ 0\leqq t\leqq 1$, denotes the well-known Peano curve covering U, then $T'': x=\phi(u),\ y=\psi(u),\ z=0,\ (u, v)\in Q$, has the same graph as J and T, but is a completely different surface. Indeed we shall mention that the "areas" of T, T', T'' are different numbers, namely 1, k, 0, respectively.

Actually there are cases where our intuition associates to different mappings (or surfaces) the same entity (as for instance T and T' above with $k=1$). Indeed, various concepts of "equivalence" have been taken into consideration, as the Lebesgue, Fréchet equivalences (for instance T and T' above for $k=1$ are Lebesgue equivalent). Classes of equivalent mappings are then denoted as Lebesgue surfaces, Fréchet surfaces. The definitions are similar to the ones which hold for curves [II] and we prefer here not to insist on these concepts.* Surfaces, or mappings $S=(T, A)$ for which each point $p\in T(A)$ has only one counterimage are said to be *simple* (and if T^{-1} is continuous then T is a *homeomorphism* between A and $T(A)$). If, for every $p\in T(A)$, the set $T^{-1}(p)\subset A$ is connected, then T is said to be *monotone*; if $T^{-1}(p)\subset A$ is totally disconnected, then T is said to be *light*. For instance, surfaces of the type

$$T: x=u,\ y=v,\ z=z(u, v),\ (u, v)\in Q,$$

that is, $T: x=z(x, y)$, $(x, y)\in Q$, are said, slightly improperly, to be "nonparametric surfaces," and they are certainly simple. The mapping $T: x=u,\ y=v$, $z=(1-u^2-v^2)^{1/2}$, $(u, v)\in Q=[u^2+v^2\leqq 1]$, whose graph is a "halfsphere" is nonparametric and simple. The mapping $T: x=2(r-r^2)^{1/2}\cos\theta,\ y=2(r-r^2)^{1/2}\sin\theta$, $z=2r-1$, where $(u, v)\in Q$, $r\cos\theta=u$, $r\sin\theta=v$, whose graph is the whole sphere $x^2+y^2+z^2=1$, is monotone, but not light since the point $p=(0, 0, 1)$ is the image of all points $(u, v)\in Q$*. The mapping $T: x=1-u^2,\ y=u-u^3,\ z=v,\ (u, v)\in A=[-2\leqq u\leqq 2, 0\leqq v\leqq 1]$, whose graph is a portion of a cylinder with generatrices parallel to the z-axis, is light, but not monotone.

Of particular interest are the "flat surfaces" or "plane mappings," *i.e.*, those mappings whose graph is contained in a plane. For instance, if τ_r, $r=1, 2, 3$, denote the projections of E_3 onto the yz, zx, xy coordinate planes, say E_{21}, E_{22}, E_{23}, then $T_r=\tau_r T$, $r=1, 2, 3$, are plane mappings, namely,

* Lamberto Cesari, Surface Area, Princeton, 1956. This book will be quoted as [SA].

$$
\begin{aligned}
&T_1\colon\ x = 0 && y = y(u, v), && z = z(u, v), && (u, v) \in A,\\
(1.2)\qquad &T_2\colon\ x = x(u, v), && y = 0, && z = z(u, v), && (u, v) \in A,\\
&T_3\colon\ x = x(u, v), && y = y(u, v), && z = 0, && (u, v) \in A.
\end{aligned}
$$

It was pointed out already by S. Banach and G. Vitali that properties of a mapping T (for instance the finiteness of the "area") have none, or very little bearing on the properties of the single functions $x(u, v)$, $y(u, v)$, $z(u, v)$, but they have an essential bearing on the properties of the pairs of functions (y, z), (z, x), (x, y), *i.e.*, on the plane mappings T_r, $r = 1, 2, 3$, above. For instance, no matter which continuous function $\phi(u, v)$, $(u, v) \in A$, we consider, the mapping $T\colon x = y = z = \phi(u, v)$, $(u, v) \in A$, has a graph completely contained in the straight line $x = y = z$ in E_3, and its "area" is zero. For nonparametric surfaces $T\colon x = u$, $y = v$, $z = z(u, v)$, all properties of T are, of course, reflected into properties of the single function $z(u, v)$.

2. Lebesgue area. For surfaces (mappings) $S = (T, A)\colon p = p(w)$, $w \in A$, $p = (x, y, z)$, $w = (u, v)$, defined by functions $x(u, v)$, $y(u, v)$, $z(u, v)$ which are continuous in A with their first partial derivatives, it is usual to assume for the area of S the value of the integral (area integral)

$$
(2.1)\qquad I(T, A) = (A^0)\int |J|\, dudv = (A^0)\int (J_1^2 + J_2^2 + J_3^2)^{1/2} dudv,
$$

and for surface integral the value of

$$
(2.2)\qquad I(T, A, f) = (A^0)\int f[p(w), J(w)]\, dudv,
$$

where $J = J(w)$ is the vector Jacobian $J = (J_1, J_2, J_3)$, $J_1 = y_u z_v - z_u y_v$, *etc.*, and where $f(p, t)$ is any given function of (p, t) continuous in (p, t) for all $p = (x, y, z) \in T(A)$, and all $t = (t_1, t_2, t_3)$. In order to assure that $I(T, A, f)$ is invariant with respect to Lebesgue, or Fréchet equivalences the further condition is added

$$
(h)\colon f(p, kt) = kf(p, t) \text{ for all } k > 0,\ t \text{ and } p \in T(A).
$$

Thus for $f = |t|$, $I(T, A, f)$ is the area integral $I(T, A)$ and condition (h) is satisfied. The definitions (2.1) and (2.2) are adequate under the restrictive conditions mentioned above but not in general. To see this let us denote by $\phi(t)$, $0 \leqq t \leqq 1$, $\phi(0) = 0$, $\phi(1) = 1$, the well-known monotone nondecreasing function which is constant on each complementary interval I_j of the ternary Cantor set in $[0, 1]$. Thus $\sum |I_j| = 1$ and $\phi'(t) = 0$ almost everywhere (a.e.) in $[0, 1]$.* Then the nonparametric light mapping (surface) $S'\colon x = u$, $y = v$, $z = \phi(u)$, $(u, v) \in Q = [0, 1, 0, 1]$ should have "area" $\geqq \sqrt{2}$, while the integral $I(T, A)$ has the value 1. Analogously, the monotone mapping (surface) $S''\colon x = \phi(u)$, $y = v$, $z = 0$ (a mono-

* E. W. Hobson, The Theory of Functions of a Real Variable, Cambridge University Press, 1927, vol. I, p. 368.

tone mapping from Q into the square $U=[0\leqq x, y\leqq 1, z=0]$) should have "area" 1 (or at least 1), while the integral $I(T, A)$ has the value zero.

A definition, which has been shown to be quite adequate is the one proposed in 1900 by H. Lebesgue. In very simple words it could be said that the Lebesgue area of a surface S is the minimum limit of the elementary areas of the polyhedral surfaces approaching S. To make this a precise definition a few more words are necessary.

We shall denote as a figure F any finite sum of disjoint closed polygonal regions in E_2 (for instance, a square, a polyhedral region). A mapping $(P, F): x=p(w)$, $w\in F$, from a figure F is said to be *quasilinear*, or a *polyhedral surface* if (a) $p(w)$ is single valued and continuous in F; (b) there exists some subdivision D of F into nonoverlapping triangles t such that each component $x(u, v)$, $y(u, v)$, $z(u, v)$ of $p(w)$ is linear in each $t\in D$, *i.e.*, of the form $au+bv+c$, a, b, c constants for each t. Then P maps each $t\in D$ into a triangle $\Delta\subset E_3$, which may be degenerated into a segment, or a single point. Then by the elementary area $a(P, F)$ of (P, F) is meant the sum $a(P, F)=\sum a(\Delta)$, where $a(\Delta)$ is the usual area of the triangle $\Delta=P(t)$ and $\sum$ ranges over all $t\in D$. We shall say that a sequence $[A_n]$ of admissible sets A_n *invades* an admissible set A, if $A_n\subset A$, $A_n\subset A_{n+1}$, $A_n^0\to A^0$ as $n\to\infty$. Finally, a sequence (T_n, A_n), $n=1, 2, \cdots$, is said to be *convergent* toward (T, A) if (a) $[A_n]$ invades A; (b) $d_n\to 0$ as $n\to\infty$ where $d_n=\sup|T_n(w)-T(w)|$ for all $w\in A_n$. Thus, if $A_n\equiv A$, $n=1, 2, \cdots$, the convergence of (T_n, A) toward (T, A) is the uniform convergence in A of $T_n(w)$ toward $T(w)$, $w\in A$.

Suppose now that (T, A) is a given continuous mapping from an admissible set A and denote by γ the class of all sequences $[(P_n, F_n), n=1, 2, \cdots]$ of quasilinear mappings convergent toward (T, A). Then the *Lebesgue area* $L(T, A)$ of (T, A) is defined by

$$L(S)=L(T, A)=\inf_{\gamma}\liminf_{n\to\infty} a(P_n, F_n). \tag{2.3}$$

Of course some reader may ask why this definition is chosen instead of considering simply "the supremum, or the limit of the elementary areas of the polyhedral surfaces inscribed in S." This is due to the fact discovered by H. Schwarz and G. Peano in 1890 that such a supremum may be $+\infty$ and such a limit may not exist even with as simple a surface as a portion of "circular cylinder." ([SA], 4.2, p. 24). The definition (2.3), proposed by H. Lebesgue in 1900, is the analogue of one of the alternative definitions of Jordan length for a curve [II]. It can be proved ([SA], 5.9, p. 37) that there exists some sequence (P_n, F_n), $n=1, 2, \cdots$, convergent toward (T, A) with $\lim a_n(P_n, F_n)=L(T, A)$ as $n\to\infty$.

In case A is itself a figure (for instance a square) it is not restrictive to suppose ([SA], 6.2, p. 61) that $F_n=A$ for all n. Then γ is the class of all sequences $[(P_n, A), n=1, 2, \cdots]$ of quasilinear mappings convergent uniformly in A toward (T, A).

3. The lower semicontinuity property of Lebesgue area. The definition (2.3) has, among others, the advantage that it makes it easy to prove the lower semicontinuity of $L(S)$. This can be expressed by the following statement:

(3.i) *If* (T_n, A_n), $n=1, 2, \cdots$, *is convergent toward* (T, A), *then* $L(T, A) \leqq \liminf L(T_n, A_n)$ *as* $n \to \infty$.

We shall prove (3.i) in the case A is a given figure, $A_n = A$ for all n, and by assuming in the definition (2.3) $F_n = A$ for all n, according to the remark at the end of Section 2. Also we may assume $\lambda = \liminf L(T_n, A_n) < +\infty$, and $L(T_n, A) < +\infty$ for all n. Then, if $d_n = \max |T_n(w) - T(w)|$, we have $d_n \to 0$ as $n \to +\infty$. By the definition of $L(T, A)$ there exists some sequence (P_{nm}, A), $m=1, 2, \cdots$, of quasilinear mappings convergent toward (T_n, A) as $m \to \infty$ with $a(P_{nm}, A) \to L(T_n, A)$. Thus we have $\delta_{nm} \to 0$ as $m \to \infty$ where $\delta_{nm} = \max |P_{nm}(w) - T_n(w)|$ for all $w \in A$. For each n, there exists an integer $m = m(n)$ such that $\delta_{nm} < 1/n$, $|a(P_{nm}, A) - L(T_n, A)| < 1/n$. Now we have $|P_{nm}(w) - T(w)| \leqq |P_{nm} - T_n| + |T_n - T| \leqq \delta_{nm} + d_n < d_n + 1/n$ for all $w \in A$. If $P_n' = P_{nm}$, then the sequence (P_n', A), $n=1, 2, \cdots$, converges toward (T, A), that is, belongs to the class γ relative to (T, A), and thus, by (2.3), we have

$$L(T, A) \leqq \liminf_{n \to \infty} a(P_n', A) \leqq \liminf_{n \to \infty} [L(T_n, A) + 1/n] = \lambda.$$

4. Plane mappings, their total variation, and absolute continuity. Let (T, A): $p = T(w)$, $w \in A$, $w = (u, v)$, $p = (x, y)$, be any continuous mapping from an admissible set A of the oriented uv-plane E_2 into the oriented xy-plane E_2', *i.e.*,

$$(T, A): \; x = x(u, v), \; y = y(u, v), \; (u, v) \in A.$$

For every simple closed polygonal region $\pi \subset A$, let us consider the oriented boundary π^* and the image $C: (T, \pi^*)$ of π^*, that is, the continuous mapping (T, π^*) defined by T on π^* (restriction of T on π^*). This is a closed oriented

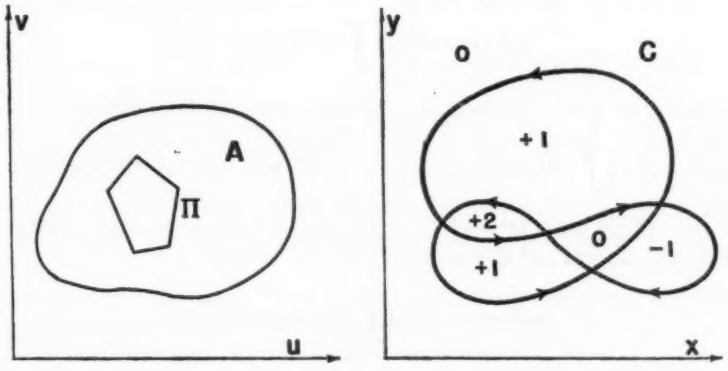


FIG. 1

curve C of the xy-plane E_2 (Fig. 1). For each point $p_0=(x_0, y_0)$ not on the graph $[C]$ of C, we can define the *topological index* $O(p_0; C)$ of C with respect to the point $p_0 \in E_2' - [C]$. Roughly speaking, $O(p_0; C)$ is the integral number of times ($\lesseqgtr 0$) in which C "links" the point p_0 in the positive direction. Suppose we assume on π^* a parameter, say s, $0 \leqq s \leqq a$. Suppose we use polar coordinates (ρ, θ) of center p_0 in E_2, then as s describes $[0, a]$, *i.e.*, w describes π^* once in the positive sense, then $p=T(w)$, describes C. The modulus $\rho=\rho(s)$ of p, *i.e.*, $\rho=\rho(s)=|p-p_0|$, is a single-valued continuous function of s on $[0, a]$ and $\rho(a)=\rho(0)$. The argument $\theta(s)$ of p is defined only mod 2π. If we fix any one of its values, say $\bar{\theta}=\theta(0)$, for $s=0$, then by continuity, $\theta(s)$ is defined on $[0, a]$ as a single-valued continuous function of s, and $\theta(a) \equiv \theta(0) = \bar{\theta} \pmod{2\pi}$. Then, by definition, $O(p_0; C) = (2\pi)^{-1}[\theta(a) - \theta(0)]$, and it is easy to prove that $O(p_0; C)$ does not depend on the parametrization of π^*, and on the choice of $\bar{\theta}$. This purely analytical definition* of $O(p_0; C)$ is certainly the most elementary one. Other equivalent and purely topological definitions are given in topology. If C is rectifiable, and we think of E_2' as the plane of the complex variable $\zeta = x+iy$, with $\zeta_0 = x_0+iy_0$, then $O(p_0; C)$ is given by the line integral

$$O(p; C) = (2\pi)^{-1} \int_C (\zeta - \zeta_0)^{-1} d\zeta.$$

The topological index $O(p; C)$ has a number of analytical and topological properties ([SA], 8.3–11, p. 85). Let us mention here that $O(p; C)$, $p \in E_2' - [C]$, is always finite, [but not necessarily bounded in E_2']; $O(p; C)$ is constant on each complementary component of the graph $[C]$ of C and is zero in the unbounded one. In the case of the illustration (Fig. 1), the values of $O(p, C)$ are given. If we assume $O(p, C)=0$ at all points $p \in [C]$, then $O(p; C)$, $p \in E_2'$, is a single-valued, integral-valued ($\lesseqgtr 0$) function of p in E_2', and $O(p; C)$ is B-measurable. Thus the L-integral

$$v(T; \pi) = (E_2') \int |O(p; C)| \, dxdy$$

exists (finite, or $+\infty$). Analogously, we may consider the numbers

$$v_+(T, \pi) = (E_2') \int O^+(p; C) dxdy \geqq 0, \qquad v_-(T, \pi) = (E_2') \int O^-(p; C) dxdy \geqq 0,$$

where $O^+ = \frac{1}{2}[|O| + O]$, $O^- = \frac{1}{2}[|O| - O]$, and $v_+ + v_- = v$. Finally, if $O(p; C)$ is L-integrable, *i.e.*, $v(T, \pi) < +\infty$, then also the number

$$u(T, \pi) = (E_2') \int O(p; C) dxdy \lesseqgtr 0,$$

exists and $|u| \leqq v$.

* P. Alexandroff and H. Hopf, Topologie, Berlin, 1935, p. 462.

If D denotes any finite system of closed nonoverlapping simple polygonal regions $\pi \subset A$, let

$$V(T, A) = \sup_D \sum_{\pi \in D} v(T, \pi),$$

and, for every point $p \in E_2'$, also

$$N(p; T, A) = \sup_D \sum_{\pi \in D} |O(p; C)|.$$

Then $N(p; T, A)$, $0 \leqq N \leqq +\infty$, is an integral-valued, single-valued function of p in E_2', and $N(p)$ is lower semicontinuous in E_2, *i.e.*, $N(p_0) \leqq \liminf N(p)$ as $p \to p_0$, for every $p_0 \in E_2'$. The function $N(p, T, A)$, $p \in E_2'$, is said to be the multiplicity function of (T, A). (It is similar to the corrected multiplicity of a real function of one real variable considered in [I], p. 239). Finally, let

$$W(T, A) = (E_2') \int N(p; T, A) dxdy.$$

Both V and W can be considered as total variations of (T, A). Indeed, V is of the type of the Jordan total variation, and W of the type of the Banach total variation of a real function of one real variable ([I], p. 321). The following basic theorem extends Banach theorem for real functions ([I], p. 321):

(4.i) *For every continuous plane mapping* (T, A) *we have* $V(T, A) = W(T, A) = L(T, A)$.

For a proof see [SA], p. 186 and p. 390. Then the common value (finite, or $+\infty$) of V, W, L can be assumed as the total variation of the plane mapping (T, A), and (T, A) is said to be of *bounded variation* (BV) if $V(T, A) = W(T, A) < +\infty$. Analogously let us put

$$V_+(T, A) = \sup_D \sum_{\pi \in D} v_+(T, \pi), \qquad V_-(T, A) = \sup_D \sum_{\pi \in D} v_-(T, \pi),$$

$$N_+(p; T, A) = \sup_D \sum_{\pi \in D} O^+(p; C), \qquad N_-(p; T, A) = \sup_D \sum_{\pi \in D} O^-(p; C),$$

$$W_+(T, A) = (E_2') \int N_+(p; T, A) dxdy, \quad W_-(T, A) = (E_2') \int N_-(p, T, A) dxdy.$$

(4.ii) *For every continuous plane mapping* (T, A) *we have* ([SA], p. 187)

$$V_+(T, A) = W_+(T, A), \; V_-(T, A) = W_-(T, A), \; V_+ + V_- = V, \; W_+ + W_- = W.$$

Thus the common values $V_+ = W_+$, $V_- = W_-$ can be assumed as the *positive* and *negative total variations* of (T, A), and the difference $\mathfrak{V}(T, A) = V_+(T, A) - V_-(T, A)$ as the *signed*, or *relative*, total variation of (T, A).

Finally if (T, A) is BV, *i.e.*, $V(T, A) < +\infty$, then $v(T, \pi) < +\infty$ for every $\pi \in A$, and thus also the following number is defined

$$U(T, A) = \sup_{D} \sum_{\pi \in D} |u(T, \pi)| .$$

(4.iii) *For every continuous BV plane mapping* (T, A) *we have* $U(T, A) = V(T, A)$.

Let us observe finally that V is "overadditive;" *i.e.*, if $[A']$ is any finite subdivision of A into nonoverlapping admissible sets, or, more generally, $[A']$ is any finite system of nonoverlapping admissible sets $A' \subset A$, then $V(T, A) \geqq \sum V(T, A')$, and the sign $>$ may hold even in the apparently elementary case where A' and A are polygonal regions.

The considerations above show how the concept of "plane mapping of bounded variation" (BV) can be founded on topological and measure theoretical considerations. On the same basis we can introduce the corresponding concept of absolute continuity (AC).

A continuous (plane) mapping (T, A): $p = T(w), w \in A, w = (u, v), p = (x, y, z)$ from the uv-plane E_2 into the xy-plane E_2', is said to be *absolutely continuous* (AC) if both the following conditions hold:

(a) *given* $\epsilon > 0$ *there is a* $\delta = \delta(T, A, \epsilon) > 0$ *such that for each finite system* $D = [\pi]$ *of nonoverlapping simple closed polygonal regions* $\pi \subset A$ *with* $\sum |\pi| < \delta$ *we have* $\sum v(T, \pi) < \epsilon$;

(b) *for every simple closed polygonal region* $\pi \subset A$ *and finite subdivision* $[\pi']$ *of* π *into nonoverlapping simple polygonal regions* π' *we have* $V(T, \pi) = \sum V(T, \pi')$.

In (a), $|\pi|$ denotes the Lebesgue measure of the set π in E_2 (area). Condition (a) is familiar, and essentially requires that v (and thus V) is "an absolutely continuous set function." Condition (b) simply requires that V is "additive" (at least in the class of the simple polygonal regions $\pi \subset A$). Conditions (a) and (b) are independent, as has been shown by examples, and it is just their logical sum (a)+(b) which is assumed here as a definition of absolute continuity (AC). There are a number of interesting properties, each of which is a necessary and sufficient condition for a mapping (T, A) to be AC.

5. A characterization of surfaces with finite area. The basic concepts of area of a mapping (T, A), $T: A \to E_3$, (Sec. 2) and of total variation of the plane mappings (T_r, A) defined by (1.2) (Sec. 4) are connected by a basic theorem, which extends formally to Lebesgue area a known Jordan theorem for Jordan length ([II], p. 489).

(5.i) *For every continuous mapping* (T, A): $p = T(w), w \in A, w = (u, v)$ $p = (x, y, z)$, *we have*

$$(5.1) \quad V(T_r, A) \leqq L(T, A) \leqq V(T_1, A) + V(T_2, A) + V(T_3, A), \quad r = 1, 2, 3.$$

Thus $L < +\infty$ *if and only if all plane mappings* (T_r, A), $r = 1, 2, 3$, *are* BV [L. Cesari, 1942].

This theorem, in spite of its analogy with the elementary Jordan theorem for curves (see II), has been shown to have a deep topological and measure theoretical basis. The proof is given in [SA], p. 295 and consists in the process of stretching and smoothing the continuous surface $S=(T, A)$ into continuous polyhedral surfaces $S_n=(P_n, F_n)$, $n=1, 2, \cdots$, with $P_n \to T$ as $n \to \infty$, and $a(P_n, F_n) \leqq V_1+V_2+V_3$, $V_r=V(T_r, A)$, $r=1, 2, 3$.

6. Peano and Geöcze areas. Let (T, A): $p=p(w)$, $w \in A$, be any continuous mapping from $A \subset E_2$ into E_3, and let us denote by T_1, T_2, T_3 the plane mappings (1.2), which are the projections $T_r=\tau_r T$ of T on the yz, zx, xy coordinate planes E_{21}, E_{22}, E_{23}. If π denotes any simple polygonal region $\pi \subset A$, π^* the oriented boundary of π, C: (T, π^*), C_r: (T_r, π^*), $r=1, 2, 3$, the continuous oriented closed curves which are the images of π^* under T and T_r. Thus $[C] \subset E_3$, $[C_r] \subset E_{2r}$, and C_r is the "projection" of C on E_{2r}. According to Section 4 we put

$$v_r = v(T_r, \pi) = (E_{2r}) \int |O(p; C_r)| \, dp, \qquad r = 1, 2, 3,$$

$$v = (v_1, v_2, v_3), \qquad |v| = (v_1^2 + v_2^2 + v_3^2)^{1/2},$$

$$V(T, A) = \sup_D \sum_{\pi \in D} |v(T, \pi)|,$$

where the supremum is taken with respect to all finite systems D of nonoverlapping simple polygonal regions $\pi \subset A$. Thus $v(T_r, \pi) \leqq v(T, \pi)$, $V(T_r, A) \leqq V(T, A) \leqq +\infty$, $r=1, 2, 3$. The number V can be thought of as an "area" of the surface $S=(T, A)$. A variant of this definition is the following one. Denote by α any plane in E_3, τ_α the projection of E_3 onto α, $T_\alpha=\tau_\alpha T$ the projection of T on α, C_α: (T_α, π^*) the oriented continuous closed curve which is the image of π^* under T_α, $[C_\alpha] \subset \alpha$. Then put $v(T_\alpha, \pi)=(\alpha) \int |O(p; C_\alpha)| \, dp$, $v^*(T, \pi) = \sup_\alpha v(T_\alpha, \pi)$,

$$P(T, A) = \sup_D \sum_{\pi \in D} v^*(T, \pi) = \sup_D \sum_{\pi \in D} \sup_\alpha v(T_\alpha, \pi).$$

Also, $P(T, A)$, like $V(T, A)$ and $L(T, A)$, is an "area" of T, in the sense that all three definitions (just as many others) are precise mathematical formulations of concepts, which all strongly appeal to our intuition as areas. As a matter of fact the following theorem has been proved:

(6.i) *For every continuous mapping (surface)* (T, A) *we have* $L(T, A)=V(T, A)=P(T, A)$ [C. B. Morrey, T. Rado, L. Cesari, J. Cecconi].

The common value of the three numbers, L, V, P, is defined as the *area* of (T, A), and is usually called the Lebesgue area of (T, A). The present definitions of V and P are the final result of successive refinements of a concept which was first proposed by G. Peano (1890). The process of successive refinements just mentioned was necessary in order to reach the basic identity (6.i) which did not hold for the previous somewhat crude concepts. A number of authors, among

them Z. de Geöcze, S. Banach, L. Tonelli, R. Caccioppoli, T. Rado, E. J. McShane, C. B. Morrey, J. Cecconi, L. Cesari have contributed to these refinements. By common agreement the area V is usually designated as the Geöcze area and P as the Peano area of $S=(T, A)$, so as to honor two mathematicians who in retrospect proved to have so deep an insight in the theory to come. For a direct proof of (6.i) see [SA] p. 390. If $V(T, A)<+\infty$, then $V_r(T, A)<+\infty$, $r=1, 2, 3$, and for every $\pi\subset A$, also the following numbers exist

$$u_r = u(T_r, \pi) = (E_{2r})\int O(p; C_r)dp, \qquad r = 1, 2, 3,$$

$$u = (u_1, u_2, u_3), \qquad |u| = (u_1^2 + u_2^2 + u_3^2)^{1/2},$$

$$U(T, A) = \sup_D \sum_{\pi\in D} |u(T, \pi)|,$$

where the same conventions are used as above. Then we have obviously

$$0 \leqq |u(T, \pi)| \leqq |v(T, \pi)| < +\infty, \qquad 0 \leqq U(T, A) \leqq V(T, A) \leqq +\infty$$

and it is clear that for some π and T we may well have $|u|<|v|$. Nevertheless the following theorem holds:

(6.ii) *For every mapping* (T, A) *with* $V(T, A)<+\infty$, *we have* $U(T, A) = V(T, A)$ [L. Cesari].

7. A Weierstrass-type integral. By the same blending of topological and analytical considerations, by means of which we have defined Geöcze and Peano areas, we may now define the concept of surface integral through a Weierstrass-type limit process.

Let $S=(T, A)$: $p=T(w)$, $w\in A$, $w=(u, v)$, $p=(x, y, z)$, be a given mapping of finite area and, for simplicity of exposition, let us suppose that A is closed finitely-connected Jordan region. Let $f(p, t)$ be a continuous function of (p, t), $p=(x, y, z)$, $t=(t_1, t_2, t_3)$, for all $p\in T(A)$ and real vector t satisfying the usual condition $f(p, kt)=kf(p, t)$ for all $k\geqq 0$, $p\in T(A)$, and t. For every simple polygonal region $\pi\subset A$ let us consider the vector

$$u = u(T, \pi) = (u_1, u_2, u_3), \qquad u_r = u(T_r, \pi), \qquad r = 1, 2, 3,$$

of norm $|u|=(u_1^2+u_2^2+u_3^2)^{1/2}$. If $|u|>0$ then the unit vector

$$a = a(T, \pi) = (a_1, a_2, a_3), \qquad a_r = u_r/|u|, \qquad r = 1, 2, 3, \qquad a_1^2 + a_2^2 + a_3^2 = 1,$$

can be thought of as the vector of the direction cosines of an "average normal" n to the piece of the surface S defined by T on π. If D is any finite system $D=[\pi]$ of nonoverlapping simple polygonal regions $\pi\subset A$, and for each $\pi\in D$, we denote by $\tilde{p}$ any point of $T(\pi)$, we may consider the sum

$$S = \sum_{\pi\in D} f(\tilde{p}, u) = \sum_{\pi\in D} f(\tilde{p}, a)\,|u(\pi, T)|$$

as an "approximate expression" of a Weierstrass-type integral $J(T, A, f)$ of f on the surface $S=(T, f)$. An index $\delta=\delta(D)$ measuring the "fineness" of D can be introduced in various ways. Then it can be proved that the following limit exists and is finite

$$J(T, A, f) = \lim_{\delta\to 0} S = \lim_{\delta\to 0} \sum_{\pi\in D} f(p, a) \mid u(\pi, T) \mid ,$$

using only the general hypotheses listed above (Cesari). The integral J is invariant with respect to both Lebesgue and Fréchet equivalences; *i.e.*, $J(T, A, f)$ has the same value in correspondence with Lebesgue or Fréchet equivalent mappings.

An index δ of fineness of a system D is, for instance, defined by the maximum of all following numbers: diam $T(\pi)$ for $\pi\in D$; $\mid \sum T_r(\pi) \mid$, $r=1, 2, 3$, and

$$U(T, A) - \sum_{\pi\in D} \mid u(T, \pi) \mid , \qquad U(T_r, A) - \sum_{\pi\in D} \mid u(T_r, \pi) \mid , \qquad r = 1, 2, 3.$$

8. Relation between area and area integral. Now the question presents itself: What relation is there between area and the area integral of Section 2, and between the Weierstrass-type integral J and the classical surface integral I of Section 2? As mentioned in Section 2 the finiteness of the area does not imply the existence of the partial derivatives of $x(u, v)$, $y(u, v)$, $z(u, v)$, and hence of the ordinary Jacobians $J_1=y_uz_v-y_vz_u, \cdots$. The example $S=(T, A)$: $x=y=z=\phi(u, v)$, where $\phi(u, v)$ is any continuous function with partial derivatives at no point, is typical since here $L(S)=0$ is certainly finite. Thus it is clear that $L(S)<+\infty$ implies the existence of J, but neither area integral, nor the classical integral I need exist.

Nevertheless, under the sole hypothesis of finiteness of the area, certain "generalized Jacobians" $\mathcal{J}_r(w)$, $w\in A^0$, can be defined a.e. in A^0, and for which we have $\mathcal{J}_r(w)=J_r(w)$, $r=1, 2, 3$, a.e. in A^0, whenever the functions x, y, z have ordinary partial derivatives a.e. in A^0. Of the various equivalent definitions of generalized Jacobians the following is certainly the simplest one, and holds for almost all $w_0\in A^0$:

$$\mathcal{J}_r(w_0) = \lim_{\sigma\to 0} \mathfrak{B}(T_r, q)/\mid q \mid , \qquad r = 1, 2, 3,$$

where $w_0\in A^0$, q denotes any square with sides parallel to the u and v axes, with $w_0\in q$, $q\subset A^0$, and $\sigma=\text{diam } q$. With these definitions the following theorem can be proved, which corresponds formally to a Tonelli's theorem for curves ([II], p. 492):

(8.i) *For any continuous mapping $S=(T, A)$ with $L(T, A)<+\infty$, we have*

$$L(T, A) \geqq (A^0) \int \mid \mathcal{J} \mid dudv, \tag{8.1}$$

where $\mathcal{J}=(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$. The equality sign holds if and only if the plane mappings (T_r, A), $r=1, 2, 3$, are AC.

Suppose now that $S=(T, A)$ and $f(p, t)$ are given as in Section 7. Then we have

(8.ii) *If* $L(S)<+\infty$ *and the plane mappings* (T_r, A), $r=1, 2, 3$, *are* AC, *then*

$$J(T, A, f) = (A^0)\int f[p(w), g(w)]dudv. \tag{8.2}$$

In other words, the integral J *is given as an ordinary surface integral (with generalized Jacobians) whenever the area is given by the corresponding area integral.*

In both (8.1) and (8.2) the integrals are L-integrals.

The question finally presents itself, whether any continuous surface $S=(T, A): x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v)\in A$, has a "representation" $(T^*, A): x=X(u, v)$, $y=Y(u, v)$, $z=Z(u, v)$, $(u, v)\in A$, for which the partial derivatives $X_u, \cdots, Z_v$ exist a.e. in A^0, and for which the area is given by the classical area integral. In other words, we ask whether another continuous mapping (T^*, A) exists which is Lebesgue, or Fréchet equivalent to (T, A), for which the plane mappings (T_r^*, A), $r=1, 2, 3$, are AC and for which $X_u, \cdots, Z_v$ exist a.e. in A^0. The answer is affirmative when Fréchet equivalence is considered (Cesari).

9. Fine-cyclic elements. For the sake of simplicity we shall suppose, as in Section 7, that A is a closed finitely connected Jordan region, say $A=J_0-(J_1+\cdots+J_\nu)^0$, (Sec. 1), where $0\leqq\nu<+\infty$ is the *order of connectivity* of A. Thus, for $\nu=0$, A is called a *disc*, for $\nu=1$, A is called an *annular region*. We have already mentioned in Section 1 the concepts of monotone mappings and of light mappings. We should now recall a precise characterization of the topological structure of any continuous mapping. For the purpose let us mention here that if we have a mapping $f: y=f(x)$ from a "set" A onto a set B, and a mapping $g: z=g(y)$ from B into a space C, then the composition mapping $F=gf$: $z=g[f(x)]$ from A into C, is said to be the product of f and g (in this order) and denoted by gf. Also, $F=gf$ is said to be a factorization of F into the two factors f and g. With this convention, we may state at once the factorization theorem of analytic topology, namely, that every continuous mapping, say (T, A): $p=p(w)$, $w\in A$, has a factorization $T=lm$ into two factors, a monotone mapping m, followed by a light mapping l (monotone-light factorization).

In order to understand this in the terms which are needed here, let us consider for every point $p\in T(A)$ the inverse set $T^{-1}(p)\subset A$. Since A is compact and T is continuous, the set $T^{-1}(p)$ is closed, hence its components g are continua $g\subset A$. By a "continuum" is meant, as usual, a bounded, closed, connected set, and hence even single points may be considered as "continua." Note that, if T is monotone (Sec. 1), then, for every $p\in T(A)$, the inverse set $g=T^{-1}(p)\subset A$ is just one continuum; if T is light (Sec. 1), then, for every $p\in T(A)$, all components g of $T^{-1}(p)\subset A$ are single points. In any case, for any continuous mapping (T, A), the collection $\Gamma=\{g\}$ of all components g of the set $T^{-1}(p)$,

$p \subset T(A)$, is a decomposition of A into disjoint continua $g \subset A$ (which may well be all single points of A). Γ is the collection of all maximal continua of constancy for T in A.

Also we may consider the family $\mathfrak{G}_0$ of all sets $G \subset A$ which are (1) open in A; (2) the union of continua $g \in \Gamma$, say $G = \sum g$ (*i.e.*, have the property that $g \in \Gamma$, $gG \neq 0$ implies $g \subset G$).

We may now consider the elements g of Γ as "points," say $\bar{g}$, and Γ as the "space," say $\bar{\Gamma}$, whose "points" are $\bar{g}$. In order to consider $\bar{\Gamma}$ as a space we actually should define a "topology" on $\bar{\Gamma}$, which turns out to be equivalent as defining in $\bar{\Gamma}$ the collection $\bar{\mathfrak{G}}$ of the open sets $\bar{G}$. This can be done easily by considering each set $G = \sum g$, $G \in \mathfrak{G}_0$, as the union $\bar{G} = \sum \bar{g}$ of the corresponding elements $\bar{g}$. Then $\bar{\mathfrak{G}}$ is just the family of all sets $\bar{G}$.

By means of these quite natural definitions $\bar{\Gamma}$ can be proved to be not only a topological space, but a "Peano space" (in particular compact, connected, locally connected). Now if $m: \bar{g} = m(w)$, $w \in A$, is the mapping from A onto $\bar{\Gamma}$ which maps each point $w \in g$, $g \in \Gamma$ into the point $\bar{g} \in \bar{\Gamma}$, then m is obviously monotone since for each $\bar{g} \in \bar{\Gamma}$, the set $m^{-1}(\bar{g}) = g$ is exactly the continuum $g \in \Gamma$, $g \subset A$. Finally, if $l: p = p(\bar{g})$, $\bar{g} \in \bar{\Gamma}$, is the mapping from $\bar{\Gamma}$ onto $T(A) \subset E_3$ defined by $l = Tm^{-1}$, then l is "light" since for each $p \in T(A)$, the components g of the set $l^{-1}(p)$ are all single points $\bar{g} \in \bar{\Gamma}$. While we refer to the usual expositions for more formal proofs, we note that $T = lm$ is a monotone-light decomposition of T, that $m: A \to \bar{\Gamma}$, $l: \bar{\Gamma} \to T(A)$, and that $\bar{\Gamma} = m(A)$ is the "middle space," or "hyperspace" of the decomposition (G. T. Whyburn, Amer Math. Soc. Colloq. Publ., vol. 28, 1942).

Obviously any "space" M which is homeomorphic to $\bar{\Gamma}$, may be considered as a middle space, or hyperspace, for T, since, if h is a homeomorphism of $\bar{\Gamma}$ onto M, and $m' = hm$, $l' = lh^{-1}$, then $T = l'm'$, $m': A \to M$, $l': M \to T(A)$, and m' is monotone and l' is light. Thus M is the middle space, and M can be called a *model* of $\bar{\Gamma}$. Also, for every monotone-light factorization $T = lm$ of T, $m: A \to M$, $l: M \to T(A)$, M is homeomorphic to $\bar{\Gamma}$, *i.e.*, M is a model of $\bar{\Gamma}$ (G. T. Whyburn).

If for some readers the previous considerations appear somewhat abstract, the following remark may be of help: A model M can be built in the Euclidean space E_3. If T is light, we may take for m the identity mapping and then $M = A$ is the middle space; if T is monotone, then we may take for l the identity mapping and then $M = T(A)$, *i.e.*, the graph of (T, A) is the middle space M.

For instance for the monotone mapping $T: x = u$, $y = 0$, $z = 0$, $(u, v) \in A = [0 \leqq u, v \leqq 1]$, M is a segment (an arc); for the monotone mapping $T: x = \sin \pi r \cos \theta$, $y = \sin \pi r \sin \theta$, $z = \cos \pi r$, $(u, v) \in A = [u^2 + v^2 \leqq 1]$, where $r \cos \theta = u$, $r \sin \theta = v$, $M = T(A)$ is the unit sphere in E_3; for the light mapping $T: x = u$, $y = v$, $z = z(u, v)$, $(u, v) \in A$, where $z(u, v)$ is a continuous function constant on no proper subcontinuum of A, the middle space is $M = A$.

Let $(T, A): p = p(w)$, $w \in A = [u^2 + v^2 \leqq 1]$ be the monotone mapping from the disc A into E_3 defined by $x = (2r - 1) \cos \theta$, $y = (2r - 1) \sin \theta$, $z = 0$, if $\frac{1}{2} \leqq r \leqq 1$, and by $x = y = 0$, $z = 1 - 2r$, if $0 \leqq r \leqq \frac{1}{2}$ where $r \cos \theta = u$, $r \sin \theta = v$. Then Γ is the

collection of all circles $u^2+v^2=r^2$ with $0<r\leqq\frac{1}{2}$, and of all single points $(u, v)\in A$, with $u=v=0$, or $\frac{1}{4}<u^2+v^2\leqq 1$ (Fig. 2). The middle space $M=T(A)$ is made up of the unit circle of the xy-plane and of the unit segment of the z-axis issuing from its center (a disc and a thread).

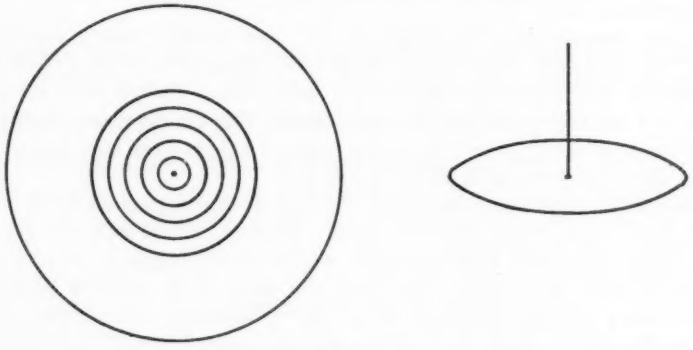


FIG. 2

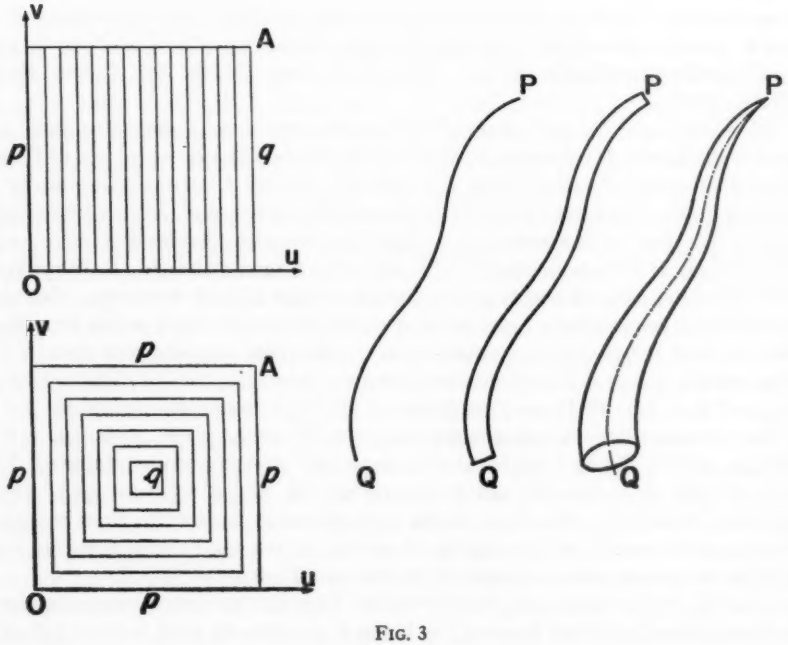


FIG. 3

Let us consider now mappings (T, A): $p = T(w)$, $w \in A = [0 \leqq u, v \leqq 1]$, from the unit square into E_3 (Fig. 3), where either Γ is the collection of all segments $g = [0 \leqq v \leqq 1,\ u = t]$, $0 \leqq t \leqq 1$, (surfaces $S_1 = (T, A)$), or Γ is the collection of the boundaries $g = [\max(|2u-1|, |2v-1|) = t]$, $0 \leqq t \leqq 1$, of all squares contained in A, concentric and similar to A (surfaces $S_2 = (T, A)$). In either case we may assume $T(w) = f(t)$, where $f(t)$ is a continuous vector function of t in $0 \leqq t \leqq 1$, constant on no subinterval of $[0, 1]$. If we suppose that $f(t)$ never takes twice the same value $f = (x, y, z)$, then T is monotone, and $M = T(A)$ is the arc PQ. The two types of surfaces S_1, S_2 apparently identical in E_3, are quite different. Surfaces S_1 can be thought as limit cases of thin strips, surfaces S_2 as limiting cases of thin cones. Two surfaces S_1, S_2, even defined by the same vector function f, are neither Lebesgue nor Fréchet equivalent.

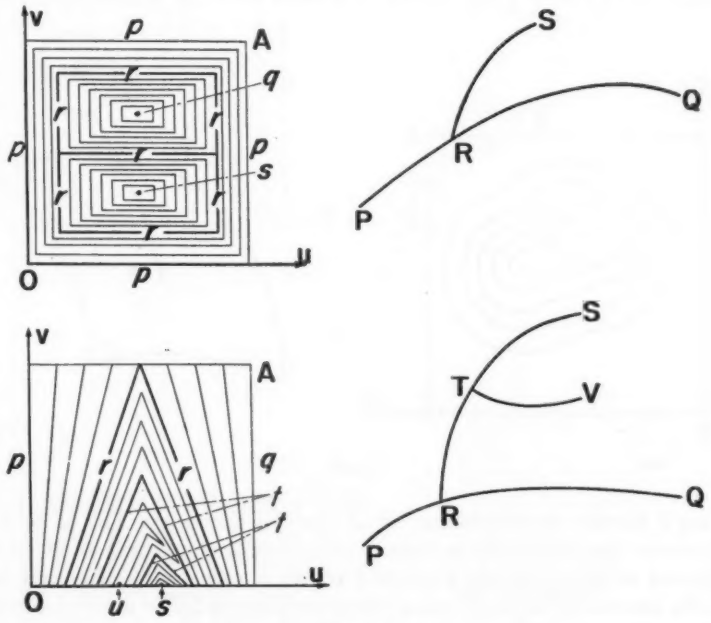


FIG. 4

Mappings (T, A) with $L(T, A) = 0$ can be characterized topologically, namely, M is a space of dimension $\leqq 1$ (T. Rado, 1945, for $\nu = 0$; R. F. Williams, 1958, for $\nu \geqq 0$). In the case A is a disc ($\nu = 0$), M has been further characterized (as a dendrite of analytic topology). In simple words, we may expect M to be a ramified system of threads, as for the simple examples given in Figure 4 for $\nu = 0$.

Mappings (T, A): $p=T(w)$, $w\in A$, with $L(T, A)>0$ must therefore possess a middle space M with some parts of dimension 2 (see, for instance, the example of Figure 2 where M is made up of a disc and a thread). These parts are important and, in the case A is a disc ($\nu=0$), they are the *cyclic elements* Σ of M. A subset Σ of M is said to be a cyclic element of M if (1) Σ is a proper continuum; (2) Σ is not disconnected by suppressing any one of the points of Σ. Any two cyclic elements Σ of M are not overlapping (they may have incommon at most one point), and the collection $\{\Sigma\}$ of all cyclic elements of M is at most countable. Thus in the example above the only cyclic element Σ of M is the disc. Again, for $\nu=0$, the cyclic elements Σ of M have been fully characterized: each of them is either a disc, or a sphere (G. T. Whyburn). The illustrations (Figs. 5, 6, 7, 8) give examples of decompositions Γ of A and of the corresponding middle spaces M of mappings (T, A) from a disc ($\nu=0$). For some only the middle space is given.

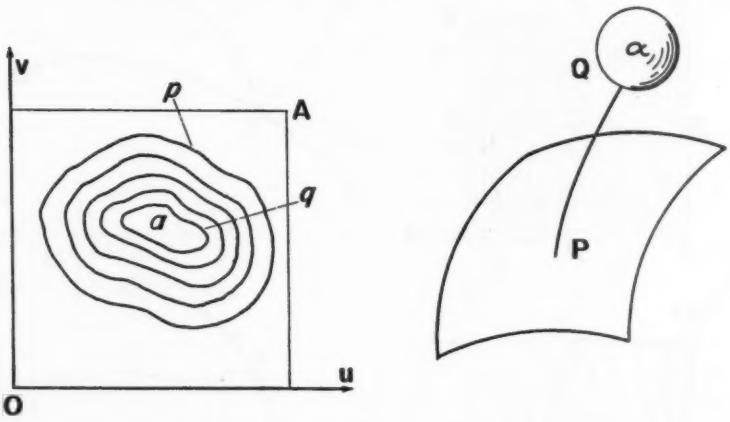


FIG. 5

For $\nu>0$ a further decomposition of M may be necessary. The parts σ of M of dimension 2 may actually be finer than the cyclic elements Σ of M, and they are denoted as the *fine-cyclic elements* σ of M. A subset σ of M is said to be a fine cyclic element of M if (1) σ is a proper continuum; (2) σ is not disconnected by suppressing any finite system of points of σ (L. Cesari, Riv. Mat. Univ. Parma, vol. 7, 1956, pp. 149–185; C. J. Neugebauer, Trans. Amer. Math. Soc., vol. 88, 1958, pp. 121–136). Any two fine-cyclic elements σ of M are not overlapping (they may have in common at most finitely many points), and the collection $\{\sigma\}$ of all fine-cyclic elements σ of M is at most countable. The illustration given here (Fig. 9) shows the decomposition Γ and the corresponding middle space M of a mapping (T, A) from an annular region ($\nu=1$). Here M constitutes a unique cyclic element $\Sigma=M$, while M contains two fine-cyclic

elements (two discs) joined by two threads. In the remaining illustrations (Figs. 10, 11, 12) other examples are given of decompositions Γ and corresponding middle spaces for mappings (T, A) from an annular region $A(\nu=1)$. For some only the middle space is given.

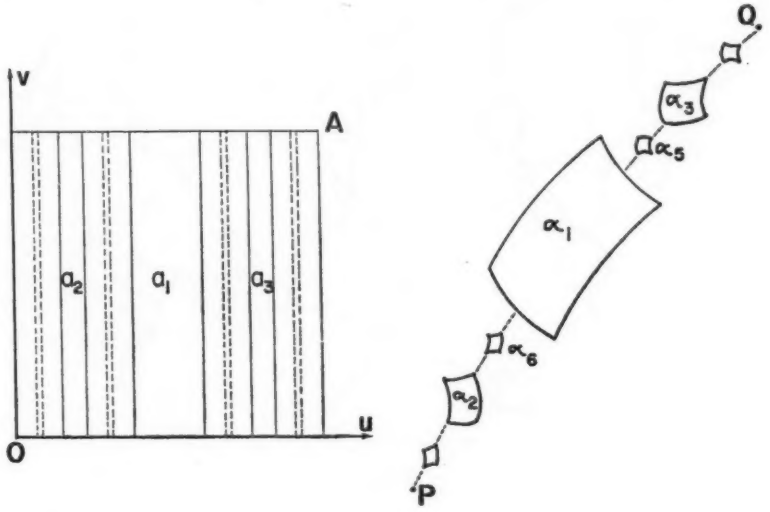


FIG. 6

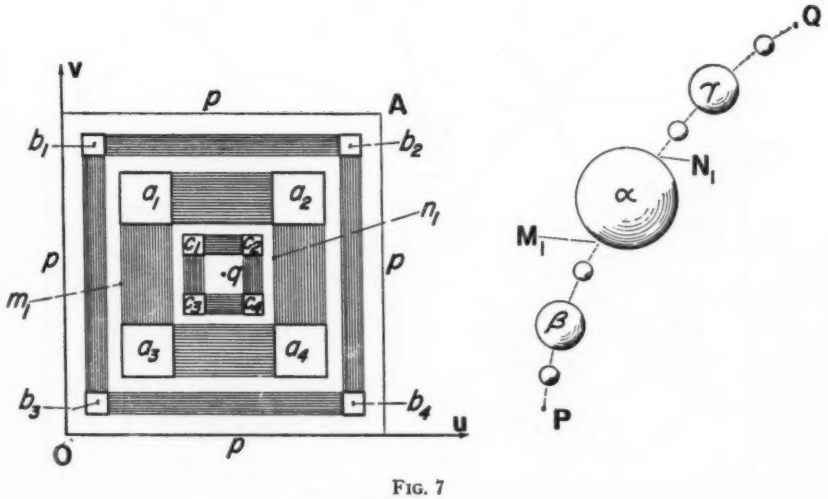


FIG. 7

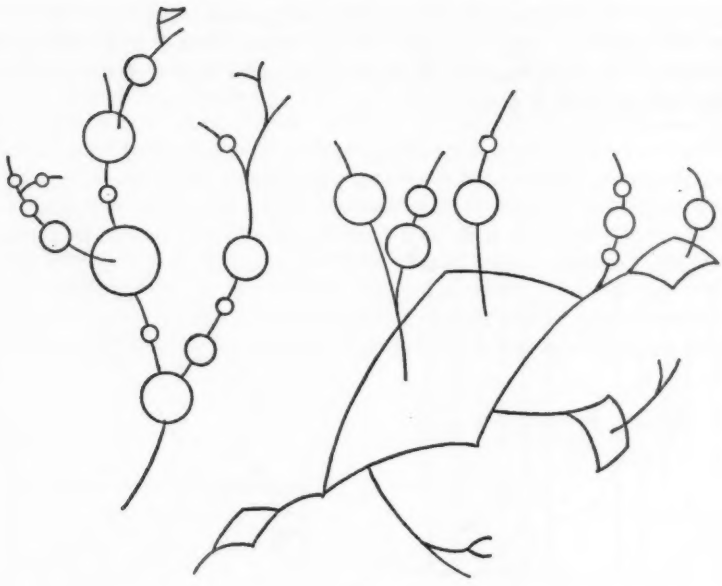


FIG. 8

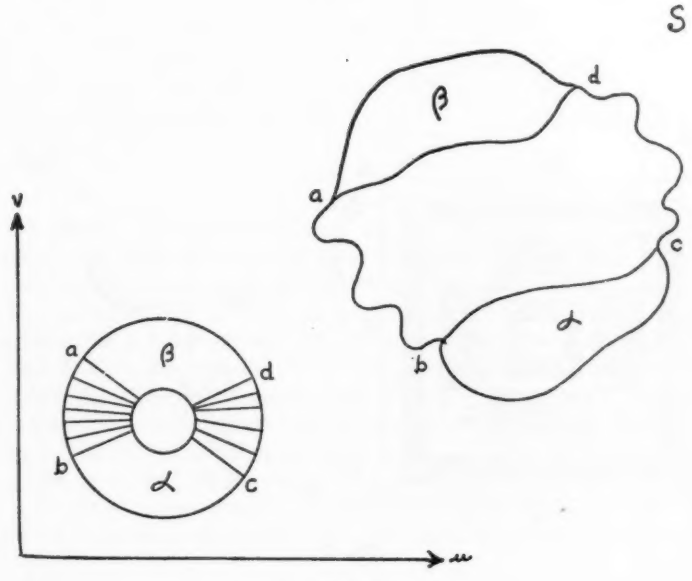


FIG. 9

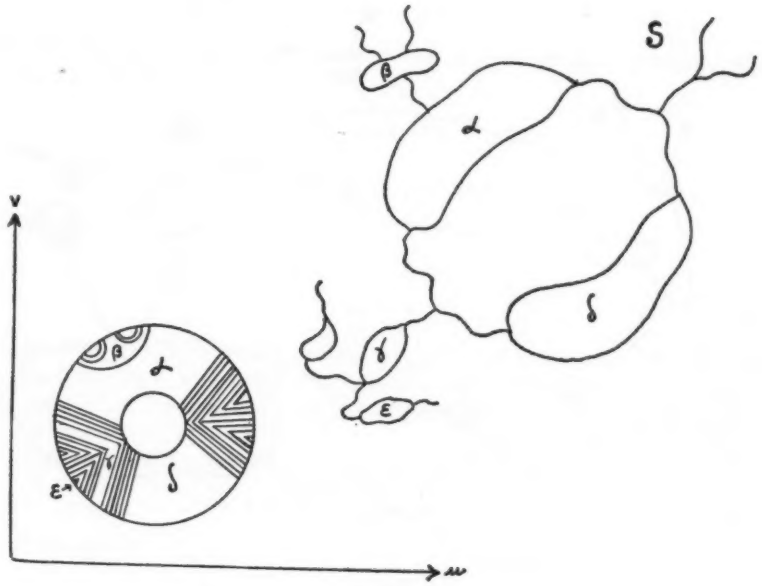


FIG. 10

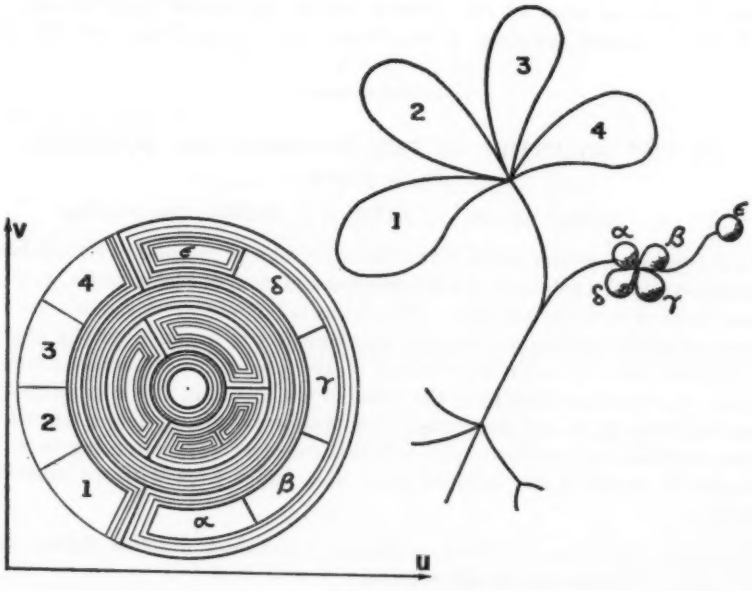


FIG. 11

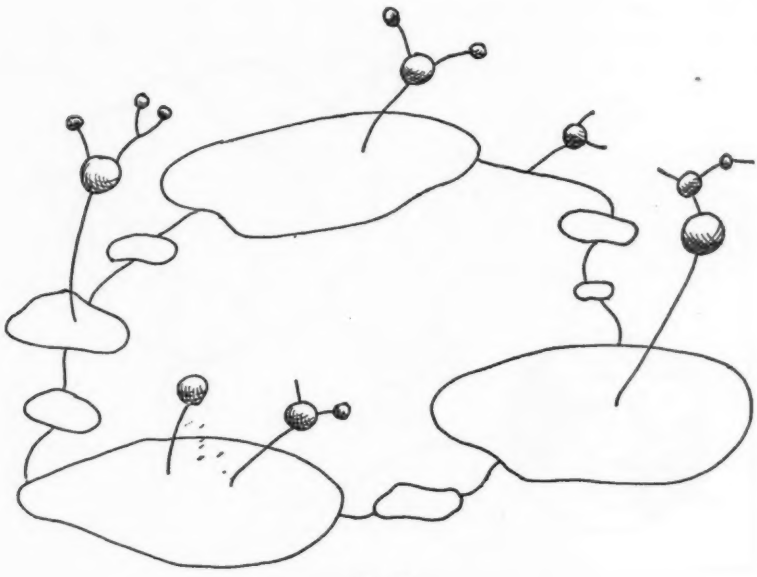


FIG. 12

On the general topic of the present article, the reader may consult [SA] and T. Rado, *Length and Area*, Amer. Math. Soc. Colloq. Publ., vol. 30, 1948.

ON THE SOLUTION OF THE DIFFERENTIAL EQUATION $f(x, y, y') = 0$

SMBAT ABIAN AND ARTHUR B. BROWN, Queens College

In this paper a procedure is given for applying the method of successive approximations to the solution of a differential equation of the type $f(x, y, y') = 0$, without solving it explicitly for y'. The theorem below asserts the existence and uniqueness of the solution under conditions weaker than those usually imposed. The theorem also contains the outline of the procedure for constructing the solution. In a corollary following the proof of the theorem, two appraisals of the remainder error at the nth stage of approximation are given; one in terms of the difference of the derivatives of the nth and $(n-1)$st approximating functions, the other in terms of the original data. The paper ends with an illustrative example.

THEOREM. *Let $f(x, y, z)$ be a continuous real-valued function defined on the closed region N determined by the relations*

$$|x - x_0| \leqq H_1, \qquad |y - y_0| \leqq H_2, \qquad |z - z_0| \leqq h_3,$$

where H_1, H_2, h_3 are positive constants, and let there be three positive constants D_1, D_2, D_3 such that, for $(x, y, z) \in N$,

$$|f(x, y_2, z) - f(x, y_1, z)| \leqq D_1 |y_2 - y_1)|, \tag{1}$$

$$|f(x_0, y_0, z_0)| < D_2 h_3, \tag{2}$$

and, if $z_1 \neq z_2$,

$$D_2 \leqq \frac{f(x, y, z_2) - f(x, y, z_1)}{z_2 - z_1} \leqq D_3. \tag{3}$$

Then there exist two positive constants $h_1 \leqq H_1$ and $h_2 \leqq H_2$ such that the differential equation $f(x, y, y') = 0$ has a unique solution $y = Y(x)$ in the interval $|x - x_0| \leqq h_1$, with $Y(x_0) = y_0$ and such that in that interval, $|Y(x) - y_0| \leqq h_2$, and $|Y'(x) - z_0| \leqq h_3$.

Furthermore, if we define k to be any constant such that

$$0 < k(h_3 D_3 + |f(x_0, y_0, z_0)|) < 2h_3, \tag{4}$$

and if we define

$$F(x, y, z) = z - kf(x, y, z) \tag{5}$$

for $(x, y, z) \in N$, then the function

$$Y_1(x) = y_0 + \int_{z_0}^{z} F(t, y_0, z_0)dt \tag{6}$$

as well as the function

$$Y_{n+1}(x) = y_0 + \int_{z_0}^{z} F[t, Y_n(t), Y_n'(t)]dt, \tag{7}$$

for every $n \geqq 1$, is well defined for $|x - x_0| \leqq h_1$, and

$$Y(x) = \lim_{n \to \infty} Y_n(x), \qquad |x - x_0| \leqq h_1. \tag{8}$$

We shall prove the theorem with the aid of the following three lemmas.

LEMMA 1. *If*

$$B = \max(|1 - kD_3|, |1 - kD_2|) \quad \text{and} \quad A = kD_1, \tag{9}$$

then

$$0 \leqq B < 1, \tag{10}$$

$$(1 - B)h_3 - k|f(x_0, y_0, z_0)| > 0, \tag{11}$$

and, for $(x, y, z)\in N$,

$$|F(x, y_2, z_2) - F(x, y_1, z_1)| \leqq A|y_2 - y_1| + B|z_2 - z_1|. \tag{12}$$

Proof. From (4) we see that $0<kD_3<2$ and since, by (3), $D_2\leqq D_3$, it follows that both $1-kD_3$ and $1-kD_2$ are less than 1 in absolute value. Hence B, as defined in (9), satisfies (10).

To prove (11), we first note that, since $D_2\leqq D_3$, the only possible values for B are $1-kD_2$ and kD_3-1. If $B=1-kD_2$, (11) follows from (2). If $B=kD_3-1$, then from (4) we have $k|f(x_0, y_0, z_0)|<h_3(2-kD_3)=h_3(1-B)$, so that again (11) is satisfied.

We now turn to the proof of (12). If (x, y, z_1) and (x, y, z_2) are two distinct points of N, then, since $k>0$, we infer from (3) that

$$1 - kD_3 \leqq 1 - k\frac{f(x, y, z_2) - f(x, y, z_1)}{z_2 - z_1} \leqq 1 - kD_2. \tag{13}$$

From (13) and (9) we obtain for $z_1\neq z_2$

$$|(z_2 - z_1) - k[f(x, y, z_2) - f(x, y, z_1)]| \leqq B|z_2 - z_1|, \tag{14}$$

a relation which is true also when $z_1=z_2$.

Now, for $(x, y, z)\in N$,

$$\begin{aligned} F(x, y_2, z_2) - F(x, y_1, z_1) &= F(x, y_2, z_2) - F(x, y_1, z_2) \\ &\quad + F(x, y_1, z_2) - F(x, y_1, z_1), \end{aligned} \tag{15}$$

which, by (5), is equal to

$$\{-k[f(x, y_2, z_2) - f(x, y_1, z_2)]\} + \{(z_2 - z_1) - k[f(x, y_1, z_2) - f(x, y_1, z_1)]\}.$$

Relation (12) is now seen to follow from (15), (1), (9) and (14). This completes the proof of Lemma 1.

LEMMA 2. *There exist two positive constants* $h_1\leqq H_1$ *and* $h_2\leqq H_2$ *such that*

$$0 < Ah_1 + B < 1, \tag{16}$$

$$h_1(|z_0| + h_3) \leqq h_2, \tag{17}$$

$$|F(x, y, z) - z_0| \leqq h_3, \tag{18}$$

for every point of the closed region $N'\subset N$ *defined by*

$$|x - x_0| \leqq h_1, \qquad |y - y_0| \leqq h_2, \qquad |z - z_0| \leqq h_3. \tag{19}$$

Proof. In view of (11) and the continuity of $F(x, y, z)$ we see that positive constants $h_1\leqq H_1$ and $h_2\leqq H_2$ exist such that for $|x-x_0|\leqq h_1$, $|y-y_0|\leqq h_2$,

$$|F(x, y, z_0) - F(x_0, y_0, z_0)| \leqq (1 - B)h_3 - k|f(x_0, y_0, z_0)|. \tag{20}$$

It is clear from (10) that h_1 can then be decreased in value, if necessary, so that

(16) and (17) are satisfied, and that (20) will remain valid. Further, if (x, y, z) satisfies (19), from the obvious inequality

$$|F(x, y, z) - z_0| \leqq |F(x, y, z) - F(x, y, z_0)| + |F(x, y, z_0) - F(x_0, y_0, z_0)| + |F(x_0, y_0, z_0) - z_0|,$$

and from (12), (19), (20) and (5), we infer that

$$|F(x, y, z) - z_0| \leqq Bh_3 + (1 - B)h_3 = h_3.$$

Hence (18) is satisfied, and Lemma 2 is proved.

LEMMA 3. *If $U(x)$ and $V(x)$ are of class C^1 on $|x-x_0| \leqq h_1$, with*

$$U(x_0) = y_0, \quad |U(x) - y_0| \leqq h_2, \quad |U'(x) - z_0| \leqq h_3, \tag{21}$$

$$V(x_0) = y_0, \quad |V(x) - y_0| \leqq h_2, \quad |V'(x) - z_0| \leqq h_3, \tag{22}$$

then

$$|F[x, U(x), U'(x)] - F[x, V(x), V'(x)]| \leqq (A|x - x_0| + B)\left(\max_{t\in[x,x_0]} |U'(t) - V'(t)|\right). \tag{23}$$

Proof. From (12) we infer that for $|x-x_0| \leqq h_1$,

$$|F[x, U(x), U'(x)] - F[x, V(x), V'(x)]| \leqq A|U(x) - V(x)| + B\left(\max_{t\in[x,x_0]} |U'(t) - V'(t)|\right). \tag{24}$$

Since $U(x_0) = V(x_0)$ we have, obviously,

$$U(x) - V(x) = \int_{x_0}^{x} [U'(t) - V'(t)]dt,$$

which, in view of (24), yields (23). Hence Lemma 3 is valid.

With the aid of the lemmas proved above, we can now prove the theorem. As a first step in establishing the existence of a solution, we introduce (6) and (7), and shall show that for $n \geqq 1$ and for $|x-x_0| \leqq h_1$, $Y_n(x)$ is well defined and of class C^1 and that

$$Y_n(x_0) = y_0, \quad |Y_n(x) - y_0| \leqq h_2, \quad |Y_n'(x) - z_0| \leqq h_3. \tag{25}$$

From (6) we infer that for $|x-x_0| \leqq h_1$, $Y_1(x)$ is well defined and of class C^1 and that $Y_1(x_0) = y_0$. Moreover from (6), (18) and (17) we see that for $|x-x_0| \leqq h_1$,

$$|Y_1(x) - y_0| \leqq h_1(|z_0| + h_3) \leqq h_2,$$
$$|Y_1'(x) - z_0| = |F(x, y_0, z_0) - z_0| \leqq h_3,$$

which implies the truth of (25) for $n=1$. The inductive proof for $n=m+1$, with $Y_m(x)$ assumed to be of class C^1 for $|x-x_0| \leqq h_1$, and properties (25) assumed for $n=m$, is exactly like that just given for $n=1$, but using (7) instead of (6). Therefore we conclude that for $|x-x_0| \leqq h_1$, $Y_n(x)$ is well defined and of class C^1, and that the relations (25) are true, for all $n \geqq 1$.

As the next step in the existence proof we shall show that, for $n \geqq 2$ and $|x-x_0| \leqq h_1$,

$$(26) \quad \max_{t \in [x, x_0]} |Y_n'(t) - Y_{n-1}'(t)| \leqq (Ah_2 + Bh_3)(A|x-x_0| + B)^{n-2},$$

$$(27) \quad |Y_n(x) - Y_{n-1}(x)| \leqq |x-x_0|(Ah_2 + Bh_3)(A|x-x_0| + B)^{n-2}.$$

From (6) and (7), by differentiation with respect to x, we obtain

$$|Y_2'(x) - Y_1'(x)| = |F[x, Y_1(x), Y_1'(x)] - F(x, y_0, z_0)|,$$

the right member of which, in view of (12) and (25), is $\leqq Ah_2 + Bh_3$. Hence (26) is true for $n=2$. Assuming (26) true for $n=m$, we obtain from (7), for $|x-x_0| \leqq h_1$, $|Y_{m+1}'(x) - Y_m'(x)| = |F[x, Y_m(x), Y_m'(x)] - F[x, Y_{m-1}(x), Y_{m-1}'(x)]|$, and comparing (21) and (22) with (25), we thus infer from Lemma 3 that

$$(28) \quad |Y_{m+1}'(x) - Y_m'(x)| \leqq (A|x-x_0| + B)\left(\max_{t \in [x, x_0]} |Y_m'(t) - Y_{m-1}'(t)|\right),$$

which, with (26) for $n=m$, implies

$$|Y_{m+1}'(x) - Y_m'(x)| \leqq (Ah_2 + Bh_3)(A|x-x_0| + B)^{m-1}.$$

Since the right member of this inequality is a monotone increasing function of $|x-x_0|$, we infer that (26) is true with $n=m+1$. Hence (26) is valid on $|x-x_0| \leqq h_1$ for $n \geqq 2$. Since $Y_n(x_0) = Y_{n-1}(x_0)$, relation (27) is obtained easily from (26) by integration of $Y_n'(t) - Y_{n-1}'(t)$ with respect to t from x_0 to x.

With (26) and (27) thus proved, we are in a position to complete the existence proof. From (16), (27) and (26) we see that the sequences $\{Y_n(x)\}$ and $\{Y_n'(x)\}$ converge uniformly on $|x-x_0| \leqq h_1$. Letting, as in (8), $Y(x) = \lim_{n \to \infty} Y_n(x)$, we have by a well-known theorem,

$$Y'(x) = \lim_{n \to \infty} Y_n'(x).$$

Since $Y_n(x)$ satisfies (25) we thus have

$$(29) \quad Y(x_0) = y_0, \quad |Y(x) - y_0| \leqq h_2, \quad |Y'(x) - z_0| \leqq h_3.$$

By differentiating each side of (7) with respect to x and letting $n \to \infty$, we obtain, in view of the continuity of $F(x, y, z)$,

$$(30) \quad Y'(x) \equiv F[x, Y(x), Y'(x)].$$

From (5) we then infer that

$$f[x, Y(x), Y'(x)] \equiv 0, \tag{31}$$

so that $y=Y(x)$ is a solution of the differential equation $f(x, y, y')=0$, valid on $|x-x_0| \leqq h_1$ and satisfying (29).

To prove the uniqueness, let $U(x)$ be any function of class C^1 on $|x-x_0| \leqq h_1$ satisfying (21) and such that $f[x, U(x), U'(x)] \equiv 0$. Then by (5)

$$U'(x) \equiv F[x, U(x), U'(x)]. \tag{32}$$

In view of (30) and (32) we obtain from Lemma 3 the relation

$$|U'(x) - Y'(x)| \leqq (A|x-x_0| + B)\left(\max_{t\in[x,x_0]} |U'(t) - Y'(t)|\right)$$

and hence

$$\max_{|x-x_0|\leqq h_1} |U'(x) - Y'(x)| \leqq (Ah_1 + B)\left(\max_{|x-x_0|\leqq h_1} |U'(x) - Y'(x)|\right).$$

Since $0<Ah_1+B<1$ we infer that $U'(x)-Y'(x)\equiv 0$, and since $U(x_0)=Y(x_0)$, we have $U(x)\equiv Y(x)$. Hence $Y(x)$ is the only function of class C^1 on $|x-x_0| \leqq h_1$ which satisfies (29) and (31). This completes the proof of the theorem.

We now give two appraisals of the remainder error.

COROLLARY. *With* $|x-x_0| \leqq h_1$,

$$|Y_n(x) - Y(x)| \leqq \frac{|x-x_0|(A|x-x_0| + B)}{1 - A|x-x_0| - B}\left(\max_{t\in[x,x_0]} |Y_n'(t) - Y_{n-1}'(t)|\right), \tag{33}$$

$$|Y_n(x) - Y(x)| \leqq \frac{|x-x_0|(Ah_2 + Bh_3)(A|x-x_0| + B)^{n-1}}{1 - A|x-x_0| - B}, \tag{34}$$

where A and B are given by (9).

Proof. Since, for $|\xi-x_0| \leqq h_1$,

$$Y'(\xi) - Y_n'(\xi) = [Y_{n+1}'(\xi) - Y_n'(\xi)] + [Y_{n+2}'(\xi) - Y_{n+1}'(\xi)] + \cdots,$$

and since the right member of (28) is a monotone increasing function of $|x-x_0|$, we infer from (28) and the formula for the sum of a geometric series, the relation

$$|Y'(\xi) - Y_n'(\xi)| \leqq \frac{A|\xi-x_0| + B}{1 - A|\xi-x_0| - B}\left[\max_{t\in[\xi,x_0]} |Y_n'(t) - Y_{n-1}'(t)|\right].$$

Since $Y(x_0)=Y_n(x_0)$, (33) then follows at once by integration of $Y'(\xi)-Y_n'(\xi)$ with respect to ξ from x_0 to x. Relation (34) follows easily from (26) and (33).

Remark. If the hypotheses of the theorem are satisfied except that (3) is replaced by a similar relation with D_2 and D_3 both negative, we can obtain a relation of the form (3) by changing the sign of $f(x, y, z)$. Furthermore, a relation of the form (3) can always be obtained if f is of class C^1 and $\partial f(x_0, y_0, z_0)/\partial z \neq 0$.

Example. Consider the problem of finding a solution of the differential equation

$$f(x, y, y') \equiv -(y')^2 + y' + 4y - 2x - \frac{7}{64} = 0 \tag{35}$$

passing through the point (0, 0). Here $f(x, y, z) = -2x+4y-z^2+z-a$, where for convenience we have written a for 7/64. We take the closed region N as defined by $|x| \leqq H_1 = 1/384$, $|y| \leqq H_2 = 1/384$, $|z| \leqq h_3 = 1/4$, so that $x_0 = y_0 = z_0 = 0$. By taking $D_1 = 4$, $D_2 = 1/2$, $D_3 = 3/2$, and $k = 1$, the inequalities (1), (2), (3), and (4) will be satisfied.

By (5) we have $F(x, y, z) = 2x-4y+z^2+a$. With $h_1 = H_1$ and $h_2 = H_2$ we verify that (16), (17), and (20) are satisfied so that $N' = N$. Since $F(t, 0, 0) = 2t+a$, by (6) we find

$$Y_1(x) = \int_0^x (2t + a)dt = x^2 + ax$$

and $Y_1'(x) = 2x+a$. Next we find $Y_2'(x) = F(x, Y_1, Y_1') = 2x+a^2+a$ and consequently

$$Y_2(x) = \int_0^x (2t + a^2 + a)dt = x^2 + (a^2 + a)x.$$

Proceeding by induction, we obtain

$$Y_n(x) = x^2 + [(\cdots((a^2 + a)^2 + a)^2 \cdots + a)^2 + a]x.$$

Remembering that $a = 7/64$, it is easy to verify that, as $n \to \infty$, the coefficient of x in the above expression for $Y_n(x)$ tends to 1/8, and therefore $Y(x) = x^2 + x/8$ is the unique solution of the differential equation $(y')^2 - y' - 4y + 2x + 7/64 = 0$ in the interval $|x| \leqq 1/384$, satisfying $Y(0) = 0$ and such that in that interval $|Y(x)| \leqq 1/384$ and $|Y'(x)| \leqq 1/4$.

Discussion. Setting $x = y = 0$ in (35) and solving for y', we obtain the values $y' = 1/8$ and $y' = 7/8$, indicating the possibility of two solutions through the point (0, 0) with different slopes at that point. It would have been natural, in applying our iteration method, to choose $z_0 = 1/8$ to get the solution for which $y' = 1/8$ at (0, 0). However, in our illustration we have deliberately avoided $z_0 = 1/8$ and instead have taken $z_0 = 0$ in order better to illustrate our method. The same solution could have been obtained by choosing any other value for z_0 sufficiently close to 1/8.

Similarly (with appropriate changes in the domain of definition of $F(x, y, z)$), if we had chosen $z_0 = 7/8$ (or a value sufficiently close to $7/8$) we would have obtained the other solution, $y = x^2 + 7x/8$.

Reference

T. H. Hildebrandt and Lawrence M. Graves, Implicit functions and their differentials in general analysis, Trans. Amer. Math. Soc., vol. 29, 1927, pp. 127–153 (for related ideas in a more general setting).

PROUHET'S 1851 SOLUTION OF THE TARRY-ESCOTT PROBLEM OF 1910

E. M. WRIGHT, University of Aberdeen, Scotland

In what follows all small latin letters denote rational integers. We take $k \geqq 1$ and $j \geqq 2$ and consider the $k(j-1)$ simultaneous equations

$$\sum_{i=1}^{s} a_{i1}^{h} = \sum_{i=1}^{s} a_{i2}^{h} = \cdots = \sum_{i=1}^{s} a_{ij}^{h} \qquad (1 \leqq h \leqq k). \tag{1}$$

A solution of the equations is called nontrivial if no set $\{a_{iu}\}$ $(1 \leqq i \leqq s)$ is a permutation of another set $\{a_{iv}\}$. We may consider (i) the problem of determining $P(k, j)$, the least value of s for which a nontrivial solution of (1) exists and (ii) that of finding an actual solution of (1) for given k, j and s. Any solution of the second problem provides an upper bound for $P(k, j)$ and so may contribute towards a solution of the first. It is known [1] that $P(k, j) \geqq k+1$ and [7] that

$$P(k, j) \leqq \tfrac{1}{2}(k^2 + 3) \ (k \text{ odd}), \qquad P(k, j) \leqq \tfrac{1}{2}(k^2 + k + 2) \ (k \text{ even}).$$

These upper bounds were obtained by an enumerative argument and not by finding an actual solution. From actual solutions, it is known [2, 3] that

$$P(k, 2) = k + 1 \ (2 \leqq k \leqq 9), \qquad P(k, j) = k + 1 \ (k = 2, 3, 5).$$

The theorem that $P(k, 2)$ exists and that $P(k, 2) \leqq 2^k$ is usually ascribed to Tarry and Escott, who published papers on the subject *circa* 1910–12 (see [2]). In fact, Prouhet in 1851 gave a solution of (1) for general k and j with $s = j^k$ and so showed that $P(k, j) \leqq j^k$. It is, of course, notorious that the names given to many mathematical problems and theorems are not those of their discoverers. But Prouhet seems to have been particularly unlucky. He discovered the first general result in this field in 1851. Although sufficient papers were written on the topic to oblige Dickson in 1920 to devote to it a whole chapter of his History [2], Prouhet's result was not completely rediscovered until 1948 [4]. But the whole subject is usually given the name of two mathematicians who rediscovered

the special case $j=2$ some 60 years after Prouhet discovered the general case. Of course, both made other contributions to the subject.

Prouhet's note [5] is no more than an "Extrait par l'auteur" of a "Mémoire présenté" to the Academy. The secretaire-archiviste of the Academy was kind enough to inform me that the memoir was returned to the author in 1852. But there appears to be no trace in the literature of its publication. The extract gives a correct rule for constructing a solution to (1) from any consecutive j^{k+1} members of an arithmetic progression. If we ignore the trivial generalisation to a progression, the following is equivalent to Prouhet's result:

Express each n $(0 \leqq n \leqq j^{k+1}-1)$ *as a "decimal" in the scale of* j. *If the least positive residue to modulus* j *of the sum of the digits of* n *in this scale is* v, *assign* n *to the set* S_v. *Then each of the sets* S_v *contains just* j^k *members, which may be taken as* $a_{1v}, a_{2v}, \cdots$ *to satisfy* (1) *with* $s=j^k$.

Prouhet did not publish a proof. But the achievement lies in finding the result and it is difficult to see how it could be found except by proof. Certainly the more general parametric result found independently by Lehmer [4] in 1948 is easy to prove (see, for example [8]) once one knows the result.

The reference to Prouhet in [2] gives his numerical example for $k=2$, $j=3$ and then goes on:—"As a generalisation, it is stated that there are numbers separable into j sets each of j^k terms, such that the sum of the hth powers of the terms is the same for all the sets when $h \leqq k$." This understatement appears to have misled subsequent writers.

The ideas of a recent article in this MONTHLY [6] can be extended to give a direct proof of Prouhet's result. As in [6], for any nonnegative integer b, we write $E(b)$ for the operator which maps any polynomial $\phi(x)$ into $\phi(x+b)$. The $E(b)$ form a subset of a commutative ring with unity; also $E(a)E(b)=E(a+b)$. For any positive integer n, let $v=v(n)$ be the least positive residue (mod j) of the sum of the digits of n, when n is expressed in the scale of j. Let ρ be any jth root of unity other than 1. If we write

$$\Lambda_t = 1 + \rho E(j^t) + \rho^2 E(2j^t) + \cdots + \rho^{j-1} E(j^t(j-1)),$$

we have

$$\prod_{t=0}^{k} \Lambda_t = \sum_{n=0}^{j^{k+1}-1} \rho^{v(n)} E(n). \tag{2}$$

If $\phi(x)$ is a polynomial whose term of highest degree is αx^m, the coefficient of x^m in $\Lambda_t \phi(x)$ is

$$\alpha(1 + \rho + \rho^2 + \cdots + \rho^{j-1}) = 0.$$

Hence Λ_t reduces the degree of any polynomial by one and the left-hand side of (2) maps any polynomial of degree not greater than k onto 0. Thus, if we operate with (1) on x^h $(0 \leqq h \leqq k)$ and put $x=0$, we have

$$(3) \qquad \sum_{n=0}^{j^{k+1}-1} \rho^{v(n)} n^k = 0, \text{ i.e., } \sum_{v=1}^{j} \rho^v H_v = 0,$$

where $0^0=1$, $H_v=\sum a_{iv}^k$, and $\{a_{iv}\}$ is Prouhet's set S_v of those n which satisfy $0 \leqq n \leqq j^{k+1}-1$, $v(n)=v$.

If we write $G_v=H_v-H_j$ $(1 \leqq v \leqq j-1)$ and observe that $\rho+\rho^2+\cdots+\rho^j=0$, we have

$$(4) \qquad \sum_{v=1}^{j-1} \rho^v G_v = 0$$

by (3). If τ is a primitive jth root of unity, (4) is true for $\rho=\tau^m$ $(1 \leqq m \leqq j-1)$ and so we have

$$\sum_{v=1}^{j-1} \tau^{mv} G_v = 0 \qquad (1 \leqq m \leqq j-1).$$

The determinant of the coefficients of the G_v in these equations is of Vandermonde type in the distinct numbers $\tau, \tau^2, \cdots, \tau^{j-1}$. Hence this determinant does not vanish and $G_1=G_2=\cdots=G_{j-1}=0$. This is Prouhet's result.

References

1. L. Bastien, Sphinx-Oedipe, vol. 8, 1913, pp. 171–172.
2. L. E. Dickson, History of the Theory of Numbers, vol. 2, Washington, 1920, Ch. 24.
3. A. Gloden, Mehrgradige Gleichungen, Groningen, 1944.
4. D. H. Lehmer, The Tarry-Escott problem, Scripta Math., vol. 13, 1947, pp. 37–41.
5. E. Prouhet, C. R. Acad. Sci., Paris, vol. 33, 1851, p. 225.
6. J. B. Roberts, A curious sequence of signs, this MONTHLY, vol. 64, 1957, pp. 317–322.
7. E. M. Wright, Equal sums of like powers, Bull. Amer. Math. Soc., vol. 54, 1948, pp. 755–757.
8. ———, Equal sums of like powers, Proc. Edinburgh Math. Soc., vol. 8, 1949, pp. 138–142.

A LOOK AT MATHEMATICAL COMPETITIONS

R. CREIGHTON BUCK, University of Wisconsin

1. Introduction. On a Thursday morning last spring, some 80,000 high school students in the United States and Canada sat down to match themselves against the First National Contest in High School Mathematics, sponsored by the Mathematical Association of America and the Society of Actuaries. Such a contest is not without precedent. In some countries, such competitions have a long tradition. One need only recall, for example, the Eötvös Competition held annually in Hungary from 1894 until 1928, and the extensive system of Mathematical Olympiads in the Soviet Union. In the United States, there have been many contests that operate on a regional basis, some sponsored by an individual

college (*e.g.*, Stanford, Mt. Mary), others growing out of the activities of a particular section of the M.A.A. (*e.g.*, Metropolitan New York, Maryland-D.C.-Virginia, Wisconsin).

The present paper is chiefly a report of the experience of the Wisconsin Section, which initiated an annual high school contest in 1956. It seemed appropriate to preface this with a discussion of some of the basic principles underlying mathematical competitions, and to include some data on other contests, here and abroad. The latter is not meant to be an exhaustive survey; I have selected only a few representative contests with an eye to variety and interest.

At the heart of the expanded program of the Association is the need to reach out to students in our high schools and colleges, and bring them a glimpse of the excitement and power of mathematics. It is my belief that mathematical competitions provide one such avenue. May I hope that we will see a wider support for the contest program of the M.A.A. among those mathematicians whose central interest lies in research? This was certainly the case in Hungary; it is presently the case in the USSR. A well-designed and exploited competition, supported both by the M.A.A. and other interested organizations, could have a beneficial effect upon the mathematical atmosphere in our secondary schools. The number of participants should not be the sole criterion of success; quality must not be sacrificed to quantity, nor must awards go only to those students who show ability to handle routine calculations with fantastic speed. We are looking for creative minds, and they are difficult to trap with the best of weapons.

2. Basic principles. In the past, many arguments have been offered in support of high school contests in mathematics. The following seem to be most representative:

(a) *To give official recognition by awards, presented in the name of the M.A.A., to some of the better students now taking mathematics in high schools.*

Most of those concerned with the administration of contests have found that teachers and students alike welcome the prestige value of awards for excellence in mathematics. Cash prizes, books, pins, certificates, all are effective.

(b) *To discover and encourage talented students who might otherwise escape attention.*

(c) *To lend additional motivation for some students to take more mathematics in high school.*

(d) *To encourage some students, especially those who are highly gifted and whose talents extend in many directions, to consider mathematics as a career.*

There is some indication that a properly designed examination, carrying the authority of official sponsorship by a national mathematical organization, can

in fact stimulate the imagination of students and show them unexpected directions of mathematical study. It is also possible that a hitherto unspectacular student may show surprising insight under the prod of a contest; an alert teacher may then be able to direct his energies toward higher goals.

(e) *To give a certain amount of tactful guidance to the high school curriculum by indicating the level of competence and maturity of viewpoint that can be expected from the better students.*

It is well known that the College Board examinations have influenced high school curricula; that this responsibility has been fully realized is made clear by the creation and activity of the Commission on Mathematics, under Dean Meder. A successful National Mathematics Contest may also exert considerable force; accordingly, great care must be given to the selection of questions.

In addition to these reasons, we have recently seen a further endorsement; the following is a quotation from a report of the Joint Congressional Committee on Atomic Energy: [1]

"*Listed below are a few of the major recommendations made to the committee that illustrate the types of approach which might assist in meeting these requirements*:

(1) *Establishment of a Federal mathematics scholarship award program to provide substantial cash awards to any high school graduate who passes an examination, at college-entrance level, in mathematics. . . . Chief objective of the award would be to arouse greater interest in high school mathematics as an important part of general education and an indispensable base for college-level study in science and engineering. . . .*

(2) *Earlier identification of potentially ablest students. . . . In order to recoup this loss of potential talent, it is proposed that statewide testing start at the 8th or 9th grade level, rather than the 12th. . . . Increased emphasis should also be placed on intellectual rigor for the ablest students and challenging programs provided.*"

There is, of course, some objection to the whole idea of competitive mathematical contests. One objection that has been raised to all tests of this sort, including the College Board's, is that in coaching the ablest students for such a contest, a teacher will neglect the education of his less able students, who are the very ones who need him the most. Even for the better students, drilling on sample test questions may not be the best form of mathematics education. Others have felt that a contest that does not emphasize the individual contestant, and which pits school against school and teacher against teacher, will emerge in the end only as a device for rating teacher efficiency. It has also been stated that success, especially where a large cash prize is at stake, is harmful both to the winners and the losers. Finally, some see a possible threat to intellectual freedom in the creation of one central committee to administer such an

influential instrument as a National Contest, tied for example into a scholarship program, and would prefer to see a larger degree of local autonomy.

Many of these criticisms have considerable merit, and are difficult to answer. The responsibility for meeting them rests with the M.A.A. and its administrative committees.

Let us now see what the five basic objectives listed above suggest about the nature of a mathematical competition. (Not unnaturally, the end product will bear some resemblance to the Wisconsin Contest!) We must first decide the level to which the test is to be directed. Several reasonable possibilities are open, and a choice between them may be determined by local conditions. Should the test be designed so that only those students who have completed $3\frac{1}{2}$ years of high school mathematics stand a chance of winning? This might be entirely defensible in a large metropolitan area, but does not seem wise if we are to think of the country as a whole. What of the gifted student who has the misfortune to attend one of the 40% of our high schools offering only two years of mathematics? It would thus seem to run contrary to objectives (b) and (c). One alternative is to give the contest at several different grade levels. There might be one test for students in grades 9 and 10, and another for students in grades 11 and 12. This indeed is the procedure followed in the Mathematical Olympiads in the Soviet Union. (See Sec. 3.) If we do not want to resort to this complexity (again a factor that may be decided by local conditions) the best solution would seem to be a single test for all grade levels, based upon the first $2\frac{1}{2}$ years work, which would have questions of sufficient originality so that it would severely tax the average senior, but would still be within the reach of a very able sophomore. (Our experience in Wisconsin has shown that these two goals are not contradictory; in last year's contest, the top 20 winners of almost 10,000 entrants consisted of 5 sophomores, 6 juniors, and 9 seniors.)

This brings up another aspect of test design. Broadly speaking an examination can be designed to test either achievement or aptitude. The first can be characterized as a timed multiple choice test which attempts to measure a wide sampling of the basic skills and concepts that make up the subject matter to be covered. Its questions are of varying difficulty, and are drawn from the more or less standard subject matter of the appropriate grade levels. Speed of performance is an important factor. ETS has designed many very effective testing instruments of this type, such as the familiar College Board Examinations and the STEP series. The 1958 M.A.A. National Contest was also of this type. At the other extreme, we find the type of test represented by the Stanford-Sylvania Competition. Here, we have an emphasis upon originality and insight rather than routine competence. The student is confronted by a handful of questions, uniformly difficult, and allowed to puzzle over them for several hours. A typical question might call for specific knowledge within the reach of those being tested, but would call for the employment of this in unusual ways requiring a high degree of ingenuity. The question may in fact introduce certain concepts which are quite unfamiliar to the student. In short, the winning student is asked to dem-

onstrate research ability. The original model for this was the Eötvös contest in Hungary. (See Sec. 4 and [6].) Here, one additional feature was noteworthy. Each test included one question that contained within it the germ of a further generalization. The contestant who discovered this, who posed for himself the more general question, and then proceeded to investigate it, was given bonus points; it is especially significant when a student does this without the guidance of a specific command on the instruction sheet. In the examinations used in the Russian Olympiads, half of the questions are of this "perceptive" or aptitude type.

Both types of tests have worthwhile features, as well as drawbacks. It is probably impossible to administer a test of the perceptive type on a large scale. For one thing, it is probably not possible to design a multiple choice type question which will allow a contestant to demonstrate the depth of insight and originality that is required; this would then impose a severe grading problem for the committee in charge. It should be noted that the Eötvös contest involved only several hundred students, that the Stanford Competition has had at most 800 entrants, and that the Olympiads seldom run as high as 700 in any one district. Our experience in grading the Wisconsin Final (1958), which was a test of the perceptive type, shows that it is feasible to handle 1000 test papers, but that there must be rather elaborate scoring procedures to ensure fairness.

In 1956, when a statewide contest was initiated, these considerations led the Contest Committee of the Wisconsin Section to adopt a two-stage examination. Originally, this consisted of two separate tests administered to the contestants on the same day, the first being chiefly multiple choice questions of the achievement type, and the second a much smaller number of questions of the perceptive type. Later, due to the increased enrollment, the two tests were given on different days, and the results on the first used as a guide in recommending entrance to the Final. (For details, see Sec. 6.) Although we have made mistakes that a more experienced group would have avoided, we have been reasonably satisfied with the results of our experience. It is natural to ask if our system can be extended to a national system. In some respects, Wisconsin reflects in miniature the problems of the national picture; there is wide variation among both its rural and urban secondary schools, accompanied by equally wide variation in ethnic and cultural backgrounds. It is my own personal belief that the pattern we have used could be adopted successfully in other Sections of the M.A.A., and integrated into the current National Contest. A participating Section would assume full responsibility for devising, administering, and grading its Final Contest; it might do the same for the more extensive Preliminary Contest, or it might find it possible to make use of a nationally prepared test for this purpose. A detailed proposal of this sort was presented at the recent Washington Conference, and has been discussed by the Committee on Contests at the summer meeting at M.I.T. However, the responsibility for implementing this, or indeed any extension of the present National Contest, must rest with the local sections; no small central committee can carry the burden alone.

The tasks that face the mathematical community are many and varied. We are asked to join in the development and implementation of a long range program to revise elementary and secondary school curricula. As college instructors, we must also help the present and future teachers of the nation to meet the challenge which this revision poses by offering them a training program suited to their needs. The colleges are also called upon to produce trained mathematicians in ever increasing numbers, and on a wide range of levels of accomplishment; this in turn requires the development of new courses and texts, both on the undergraduate and graduate level. The necessary and sufficient conditions for attaining all of these are easily stated: (1) manpower (2) money. Ultimately, leadership for much of this undertaking may come from within the active sections of the M.A.A.; the present National Contest owes its existence to one such section, that of Metropolitan New York.

3. The Mathematical Olympiads of the U.S.S.R. In the last year, a considerable amount of reliable information has been published about the educational system of the Soviet Union, and in particular about its program for the mathematically gifted. (See especially the authoritative book by A. G. Korol [5], and the article by B. V. Gnedenko in the MONTHLY [3]). Although there are other equally interesting aspects to the Olympiad system, we shall concentrate only on the Contest. Described briefly, this is a two-stage competitive examination sponsored in a local district for its secondary school pupils by an Administrative Committee from the local University or Technical Institute; only those students completing the first round are admitted to the second, and prizes are awarded to the winners of the second round. The nature of the questions used seems to vary somewhat with the particular district, but more often than not are of the perceptive, rather than achievement type. Descriptions of the Olympiads, often accompanied by the test questions, are frequently reported in *Uspehi Matematicheski Nauk*, and for the interest of readers, I have included the following free translations of several of these from Volume 11; it is not yet clear whether the very recent changes in Soviet educational policy will alter the pattern which had been set, starting with the first Olympiad in Leningrad (1934).

The Mathematical Olympiads in Lvov during 1955 and 1956

A. S. KOVANKO

During the school year 1954/55, a Mathematical Olympiad was held in Lvov for school children of the 7th, 8th, 9th, and 10th grades, based on somewhat different principles from those used in previous years. The first round of the tournament was held in the schools and the second round at the University, as before; however, the contest questions in both rounds were more of a technical nature, and thus were not similar to those usually asked on Olympiad tests in previous years. The attendance was high. 800 students took part in the first round, and 375 were admitted to the second round. Of these, 90 won prizes or certificates.

During the school year 1955/56, an Olympiad was held on March 25 for the students in the city of Lvov, sponsored by the University and by the Society for the Promulgation of Political and Scientific Knowledge. The contest was given separately for students in the 8th, 9th, and 10th grades, and consisted of only one examination. There were 114 entrants, of whom 10 were selected

as winners. The best of these was a student in the 9th grade (school No. 35) B. L. Zgelskii. The group also had certain preparatory work to do in the form of lessons and exercises. The following are some of the more interesting problems.

8th grade

1. Solve the equation $\sqrt[3]{(8-x)^2}+\sqrt[3]{(27+x)^2}=\sqrt[3]{\{(8-x)(27+x)\}}+7$.
2. Through two given points on the circumference of a circle, draw two parallel chords having a specified sum.

9th grade

1. Solve the equation

$$\sin\left(\frac{\pi}{10}+\frac{3x}{2}\right)=2\sin\left(\frac{3\pi}{10}-\frac{x}{2}\right)$$

2. Given an equilateral triangle ABC, mark a point A_1 which is $\frac{1}{3}$ of the way from B toward C, a point B_1, $\frac{1}{3}$ of the way from C toward A, and a point C_1, similarly placed on side AB. Show that the lines AA_1, BB_1, CC_1 form the sides of a triangle KLM whose area is exactly $\frac{1}{7}$ of the area of triangle ABC.
3. Take 121 terms of each of the arithmetic progressions 2, 7, 12, $\cdots$ and 2, 5, 8, $\cdots$. How many numbers will there be in common?

10th grade

1. Prove that

$$\sin x+\sin 2x+\cdots+\sin nx=\frac{\sin \frac{1}{2}nx \sin \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x}.$$

2. Solve the equation $4^x-(13)6^{x-1}+9^x=0$.
3. For a cone with altitude 12 and base of radius 4, find the inscribed cylinder of maximum surface area. (Uspehi Mat. N. 11(1956) 255–256)

The Fourth Mathematical Olympiad at Ordžonikidze

F. S. Churikov

The Olympiad was held during the school vacation of the school year 1955/56, for children in the 8th, 9th, and 10th grades. It was arranged by the North Osetinsk State Pedagogical Institute and with the assistance of the City Department of National Education. Preparation for the Olympiad was conducted in the Mathematics Circles of the Schools . . . [*Here, there follows a description of the supplementary lectures that often form a feature of the Olympiads.*] The first round of the Tournament took place on March 28. As in previous years, it was conducted in the schools, and supervised by teachers who also marked the papers. The tests, which had been prepared by the Organizing Committee, were sent to the schools on the eve of the first competition. It was held in 15 schools within the city, and 2 country schools. The second round was held on April 15 in the Institute, and was supervised by the Organizing Committee. Students were admitted to the second round who had scored at least 12 of 22 points on the first round test. The numbers taking part are given below:

	8th grade	9th grade	10th grade
round 1	36	53	92
round 2	7	15	49

Prizes were awarded to five students. In addition, six students were awarded honorable mention. [*At this point, Professor Churikov gives what seems to be the complete set of examinations used in the*

contest; I have selected a few representative questions from each part to show the range of difficulty. At each grade level, the tests in both rounds consisted of 5 questions apiece.]

8th grade, First Round

2. Construct the graph of $y=1/(x^2-4)$.
4. Given a circle, and two radii, construct a chord that is trisected by the radii.

8th grade, Second Round

1. What must be the values of p and q if the roots of $x^2+px+q=0$ satisfy the two equations

$$x_1 - 2x_2 = 2,\ 2x_1 - 3x_2 = 5?$$

5. What is the value of the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}?$$

9th grade, First Round

2. Construct the graph of $y=1/(\sqrt{x}-1)$.
4. Find the sum of the numbers

$$1 + 11 + 111 + \cdots + \underbrace{111\cdots 111}_{n \text{ times}}.$$

9th grade, Second Round

1. Let a, b, and c be consecutive terms of an arithmetic progression. Show that this is also true of the numbers a^2+ab+b^2, b^2+bc+c^2, and c^2+ac+a^2.

3. Show that (log 2)(log 5) = .2104, without using tables to find log 2 or log 5.

10th grade, First Round

1. Using the method of mathematical induction, prove that

$$(1!)1 + (2!)2 + (3!)3 + \cdots + (n!)n = (n+1)! - 1.$$

3. Through vertex A of the triangle ABC construct the angle bisector, the median, and the altitude. If these lines divide the angle A into four equal parts, prove that A is a right angle.

10th grade, Second Round

2. Solve the system of equations

$$(x+y+z)(ax+y+z) = k^2,$$
$$(x+y+z)(x+ay+z) = l^2,$$
$$(x+y+z)(x+y+az) = m^2.$$

3. Find the maximum and minimum values of the function $y=2\sin x-\cos 2x$.

(Uspehi Mat. N. v. 11 (1956) 251–253)

Before leaving this subject, I should like to add two remarks. Because of the objectivity of a multiple choice test, there can be valid comparisons; "passing grade is 70%" has an unambiguous meaning. However, in evaluating the effectiveness of a test such as those described above, variation can arise because of different standards of success. In particular, it would be interesting to have some valid measure of the difficulty which these questions have for Soviet students; comparative test scores or other data have not, so far as I know, been

published. (In this connection, Dr. Robert Kalin of Florida State University has made some tentative studies of the entrance examinations for admission to the mathematics department of Moscow University [4]). The second remark bears on enrollments. Using the published figures for admission to the first round, one finds totals ranging from less than 200 in smaller districts such as Ivanova and Ordžonikidze to over 1000 in Moscow and Leningrad. From the relatively small size of these, compared with estimated school enrollments, it is likely that there is considerable preselection of contestants.

4. The Eötvös prize competition in Hungary. The most complete description of this mathematical contest is to be found in the book by Jozsef Kurschak, published the year after its termination in 1928. This contains problem lists, representative solutions by contestants, and names of the winners [6]. More accessible, perhaps, is the article by Rádo published in the MONTHLY in 1932 [9]. This contest, held annually for over thirty years, was open to high school graduates who were enrolled in their first semester in college. It was held in October, and administered by college faculty locally. Each test consisted of three questions, usually selected from the areas of arithmetic, algebra and geometry, and the students were allowed to use books for reference; stress was thus laid on innate ability and insight rather than upon memorization and speed. As mentioned above, additional credit was given to those who were able to go further than the posed question. It is interesting to speculate upon the relationship between this contest and the mathematical fertility of Hungary. Among the winners of the Eötvös prize, one might mention Fejér, Karman, Haar, Pólya, Riesz, Szegö, Rádo, Rédei, and others. The following examples, taken from the Kurschak book, will illustrate the type of problem used.

1. Given a right triangle ABC, with right angle at C. Is there at least one point P inside the triangle such that $\angle PAB = \angle PBC = PCA$?

2. Find all possible whole numbers x, y, and z for which

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

is a whole number.

3. Suppose that x and y are whole numbers such that $2x+3y$ is a multiple of 17. Show that $9x+5y$ is also a multiple of 17.

4. Show that any closed curve that encloses a triangle must be longer than the perimeter of the triangle.

5. In a triangle ABC, show that $\sin(A/2)\sin(B/2)\sin(C/2) \leq 1/4$.

5. The Stanford University competitive examination. Now sponsored jointly with Sylvania Corporation, this contest has been held annually for a decade. Under the guidance of Professor Pólya, this has followed closely the pattern of the Eötvös Prize, except that it is given in the spring while students are still in high school. The questions used are of striking originality, and have always been a definite challenge to the students; only a handful of entrants produce

solutions worth examining in detail, and from these, the top dozen can be selected as winners. The test questions, accompanied by a discussion of possible solutions by Professor Pólya, have been published each year [8]. The number of entrants has risen slowly, reaching 836 in 1958. The following sample question might be considered somewhat typical.

In any triangle, the sum of the . . . is greater than the semi perimeter. Replace the dots . . . successively by (I) altitudes (II) medians (III) angle bisectors. You obtain three different assertions. Examine each: is it true or false? Prove your answer.

6. The Wisconsin section M.A.A. contest. The following description applies to the 3rd annual contest (1958). The competitive examination was separated into two stages, a Preliminary Contest held in February, and a Final Contest held in April. The former consisted of a multiple choice test, conducted in the participating high schools, and proctored and scored by the teachers. There were 18 questions of varying difficulty, based on three semesters' work. 9516 contestants from 237 schools participated. Certificates were awarded to the top scorer at each school. The complete cost of this stage was more than covered by the charge of \$1.00 per school, and \$.05 per contestant. The score distributions were reported to the central committee who determined a recommended entrance score for admission to the Final Contest; this figure was arbitrarily set to admit about 1000 to the Final. Contestants reported to 27 centers scattered over the state where they took the second stage test; this was a nonobjective perceptive type test, modeled on that in the Eötvös Prize Contest. It was proctored by college personnel, and the papers sent to Madison where they were scored by a team of graduate students, and checked by members of the Committee; 916 students entered the Final, from which 38 were selected as first, second or third prize winners, and 100 as honorable mention. The cost of the contest and prizes (\$20 for first place, together with an Association pin, and a book set for the winner's high school, lesser cash awards for second and third prize winners) was covered completely by the fee of \$1.00 charged each entrant in the Final. We quote several sample questions.

Preliminary

1. Walk one mile East, two miles North, three miles West, four miles South, five miles East, and six miles North. How far are you from your starting point?

5 mi. 7 mi. 13 mi. 21 mi.

2. Let x, y and z be three different positive numbers, with x smaller than y, and z larger than y. Which of the following is always the largest?

y/x z/x z/y none

Final

1. Four points A, B, C, and D are given on a line. Show how to construct a pair of parallel lines through A and B, and another pair of parallel lines through C and D, so that both pairs intersect to form a square.

2. Find a pair of whole numbers x and y with $11x-13y=1$, and with $x+y$ larger than 50.

7. The national M.A.A. contest, 1958. I shall assume that most readers of the MONTHLY have had an opportunity to examine the test which was used in 1958, and have some acquaintance with the history of its development. An interesting summary of this, together with an evaluation of the results of the contest, may be found in a recent article by Professors Fagerstrom and Lloyd [2]. A few comments may be helpful. The test used emphasized both speed and power; it is too long to be completed by an average contestant in the allotted time. A definite advantage is enjoyed by those with more training, but occasionally winners emerge from the lower semester students. The maximum score was 150. One student did in fact achieve a score of 146, but the median score was 31, and only 70 students of the entire 80,000 succeeded in making better than 90. A high score can be achieved only by a student who is fantastically fast, or who is able to bring his intuition to bear and consistently guess the most likely answer from those presented. Every participating school received a pin to award to its top entrant; in addition, schools received certificates if their team (top three) placed in the best 10% in their national area. Although entering schools were charged \$.20 per test, for the first twenty, and \$.10 for each additional test, the contest was not self-supporting. In addition to the awards mentioned above, local contest committees were able in some cases to secure contributions from local industries. It has been felt that it would be inadvisable to make large awards to the winners in a high school contest which is given in the high schools and proctored by the mathematics teachers; rather than reflecting an uncharitable attitude toward teachers, this comment has in fact the support of members of the N.C.T.M. who feel that this responsibility should be placed elsewhere.

8. Other local contests. It is not feasible in this report to give a detailed description of each of the multitude of high school mathematics contests held each year. In a survey made several years ago, Professor Lloyd reported on almost sixty contests; this report contains much useful comparative information [7]. Some contests are similar to the former Metropolitan New York M.A.A. Contest; among these, one might mention the contest sponsored by the Maryland-District of Columbia-Virginia section. Others seem to reflect the viewpoint of the Wisconsin Section; in this category belongs the Central Valleys Mathematics Quiz of California, sponsored jointly by a newspaper chain and a group of state colleges, and the Michigan Mathematics Prize Competition. Such variation is indeed welcome; I hope that there will be continued experimentation, until each Section of the Association is actively sponsoring a contest program best suited to its local objectives and circumstances.

9. A proposed expansion of the national M.A.A. contest. The administration of the program would rest with a Central Contest Committee, assisted by Local Contest Committees within each section of the M.A.A. Their duties and separate responsibilities are specified below.

The Contest would be a two-stage tournament. The test would be prepared by the Central Committee. The first round would be held in early March or late February.

This would be a multiple-choice test, with subject matter confined to the first two years of high school mathematics. As at present, it would be given in the high schools and proctored by teachers. There might be 25–30 questions to be answered in 80–90 minutes.

The answer sheets from the Preliminary would be sent to the Local Committee which had made the initial arrangements, and would be machine-graded, under their direction. State-wide winners, and winners within each school, then would be announced. Appropriate prizes such as pins and certificates should be awarded. The method of selection of winners and the administration of awards is left to the Local Committee; perhaps the top ten per cent of the contestants should receive recognition.

Each Local Committee sends the score distribution from its Preliminary to the Central Committee. There they are collated and a *recommended admission score for the Final* is determined. This figure is set to admit approximately four per cent of the Preliminary contestants to the Final; the limiting factor here is the number of papers that can actually be dealt with.

Using the recommended score as a guide, suitably modified to meet local conditions, the Local Committees invite schools to select contestants to enter the Final round of the tournament. The Final test should consist of *five* questions to be worked during $1\frac{1}{2}$–2 hours. They should be nonroutine perceptive-type problems that call for unusual insight. The Final contest should be given at a number of centers within each local region, usually at colleges or universities. It is hoped that this could be done in conjunction with lectures, displays, expositions, *etc.*

The responsibility for scoring the Final rests with the Local Committee. The Central Committee will have supplied suggested solutions and scorings, but the judges must be on the alert for an unorthodox approach. All persons involved in the judging should be paid.

The Local Committees send their score distributions and the top three per cent of their papers to the Central Committee. There they are collated, re-examined for uniformity of grading, and winners selected. There might be 100–200 National Winners; the number would be determined by the number of prizes available. These are announced both by the Central Committee and by the appropriate Local Committees; there should also be some process for honoring the school and teacher responsible for helping produce a National Winner.

The cost of the contest should be carried by a *small* fee charged each school, together with a *small* per-student fee. Money for prizes would come from industry or government sponsorship.

References

1. Joint Committee on Atomic Energy, Can we continue to ignore these warnings? Math. Teacher, vol. 51, 1958, pp. 240–245.

2. W. H. Fagerstrom and D. B. Lloyd, The national high school mathematics contest, Math. Teacher, vol. 51, 1958, pp. 434–439.

3. B. V. Gnedenko, Mathematical education in the USSR, this MONTHLY, vol. 64, 1957, pp. 389–408.

4. Robert Kalin and Robert Fouch, Mathematical testing in Russia, Math. Teacher, vol. 50, 1957, pp. 181–182, 236–240.

5. A. G. Korol, Soviet Education for Science and Technology, New York, 1957.

6. Jozsef Kurschak, Matematikai Versenytetelek, (Hungarian), Budapest, 1929.

7. D. B. Lloyd, The national status of mathematics contests, Math. Teacher, vol. 49, 1956, pp. 458–463.

8. Bulletin of the California Mathematics Council, 1948–1958.

9. T. Radó. Mathematical life in Hungary, this MONTHLY, vol. 39, 1932, pp. 85–90.

CONFERENCE ON THE COMMITTEE ON THE UNDERGRADUATE PROGRAM

A conference on the work of the Association's Committee on the Undergraduate Program in Mathematics was held at the Burlington Hotel, Washington, D. C., on November 15 and 16, 1958.

The following forty-seven persons attended the conference:

Max Beberman	UICSM Mathematics Project, University of Illinois
E. G. Begle	Yale University, School Mathematics Study Group
R. H. Bing	University of Wisconsin
R. C. Buck	Stanford University
R. R. Bush	University of Pennsylvania. (American Psychological Association)
E. A. Cameron	University of North Carolina
J. W. Cell	North Carolina State College. (American Society for Engineering Education)
L. W. Cohen	University of Maryland
R. P. Dilworth	California Institute of Technology
W. L. Duren, Jr.	University of Virginia
F. A. Ficken	University of Tennessee
R. C. Fisher	Ohio State University
G. E. Forsythe	Stanford University
H. M. Gehman	University of Buffalo
A. M. Gleason	Harvard University
W. T. Guy, Jr.	University of Texas
P. G. Hoel	University of California, Los Angeles
R. D. James	University of British Columbia
J. L. Kelley	University of California, Berkeley
J. G. Kemeny	Dartmouth College
C. B. Lindquist	U. S. Office of Education
W. G. Madow	Stanford Research Institute
J. R. Mayor	American Association for the Advancement of Science
E. J. McShane	University of Virginia
L. J. Montzingo	University of Buffalo
Ivan Niven	University of Oregon
E. P. Northrop	University of Chicago
F. G. O'Brien	National Science Foundation
R. E. Paulson	National Science Foundation
G. B. Price	University of Kansas
A. L. Putnam	University of Chicago
Mina Rees	Hunter College
P. C. Rosenbloom	University of Minnesota
A. E. Ross	University of Notre Dame
R. E. K. Rourke	Commission on Mathematics
Patrick Suppes	Stanford University
H. W. Syer	Kent School
G. B. Thomas, Jr.	Massachusetts Institute of Technology
D. L. Thomsen, Jr.	Watson Scientific Laboratory, Columbia University. (Society for Industrial and Applied Mathematics)
A. W. Tucker	Princeton University
F. E. Ulrich	Rice Institute
R. J. Walker	Cornell University

A. D. Wallace	Tulane University
S. S. Wilks	Princeton University
C. R. Wylie, Jr.	University of Utah
J. W. T. Youngs	Indiana University
Mark Zemansky	City College of New York. (American Institute of Physics, and American Association of Physics Teachers)

Speakers were invited to discuss the various topics on the program, and in most cases outlines were prepared and distributed to the participants in advance of the Conference. Summaries of the prepared talks and of some of the discussion are given herewith.

Price: Purpose of the Conference. The purpose of the Conference was to assist in formulating plans and policies for the future work of the Committee on the Undergraduate Program. This Committee, formed early in 1953, proposed in 1958 that the time had come to reorganize and expand its efforts. Accordingly, at the Committee's request, it was discharged at the end of August, 1958. This Conference was called to re-examine the assignment of the Committee on the Undergraduate Program and to take steps to establish a new Committee with adequate funds, personnel, and program.

Duren: Report from the original Committee on the Undergraduate Program. The speaker reviewed the goals and activities of the original CUP since its appointment in January 1953. The primary purpose of the CUP was to act as a bridge between research and curriculum. The effort of the CUP was focused on the first year of college. No attempt was made to write text books, but source books were written. Among the primary goals of the CUP were:

1. To interest young mathematicians in teaching problems, and to encourage senior mathematicians to become more aware of the needs of teaching; to point out the importance of finding a balance between critical mathematics, and the desires and motivations of 18-year olds;
2. To contribute to institute and lecture programs;
3. To establish contact with other groups with purposes similar to those of the CUP.

No specific curriculum suggestions were made until it was known that the original CUP was to be discharged.*

The speaker suggested that in the reorganization of the CUP some thought be given to reorganizing into groups, or one large group subdivided into subgroups by locality.

Rourke: Report from the Commission on Mathematics. The speaker drew attention to the books and pamphlets that have been written and distributed by the Commission, and to the conferences that have been arranged to discuss the work of the Commission. Out of these conferences there has emerged a clarification of the proposals of the Commission, especially on the following particular points:

(1) The role of set theory in the high school curriculum;
(2) the continuing role of skills;
(3) the increasing emphasis on structure as contrasted with manipulation in algebra;
(4) the use of the word "modern."

The speaker outlined the following nine point program of the Commission for the class of college-capable students in the secondary schools:

1. Strong preparation both in concepts *and* in skills, for college mathematics at the level of calculus and analytic geometry.

* Anyone interested in the reports issued by this committee may obtain the following by writing to the Buffalo office of the Association: (1) Collected Reports of the CUP (1957) and (2) Outline of Recommended Courses (1958).

2. Understanding of the nature and role of deductive reasoning—in algebra, as well as in geometry.
3. Appreciation of mathematical structure ("patterns")—for example, properties of natural, rational, real, and complex numbers.
4. Judicious use of unifying ideas—set, variable, function, and relation.
5. Treatment of inequalities along with equations.
6. Incorporation with plane geometry of some coordinate geometry, and also essentials of solid geometry and space perception.
7. Introduction in grade eleven of fundamental trigonometry—centered on coordinates, vectors, and complex numbers.
8. Emphasis in grade twelve on elementary functions (polynomial, exponential, circular).
9. Recommendation of additional alternative units for grade twelve: *either* introductory probability with statistical applications *or* an introduction to modern algebra.

Finally, the speaker emphasized the need of appropriate teacher training programs, for prospective teachers as well as those in service. He cited cases of teachers attending university summer sessions in the hope of building up their knowledge of newer viewpoints in mathematics, only to get traditional courses of a sort that have almost disappeared from many university programs.

Discussion: The question was raised as to the desirability of teaching polynomial calculus in secondary school, and Mr. Rourke pointed to the critical shortage of qualified teachers. The problem of the certification of teachers was discussed; it was pointed out that the Commission has no authority here, but can only make recommendations. There was considerable feeling that the Mathematical Association of America should likewise formulate a set of strong recommendations.

Beberman: Report on the University of Illinois Committee on School Mathematics. The speaker stated that the aim of the UICSM is to develop a four-year program of high school mathematics that develops understanding as well as manipulative skills. The program is aimed primarily at the college-capable student. At present there are about 55 schools in 20 states using the UICSM materials as pilot schools.

The units and grade level which comprise the UICSM program are:

Unit	*Grade Level*	*Descriptive Title*
1	9th Grade	Arithmetic of real numbers.
2	9th Grade	Pronumerals, generalizations, manipulation.
3	9th Grade	Equations, inequations, applications.
4	9th Grade	Ordered pairs, graphs.
5	10th Grade	Relations, functions.
6	10th Grade	Geometry.
7	11th Grade	Real number system, induction.
8	11th Grade	Exponents, logarithms.
9	11th Grade	Complex numbers, systems of quadratics.
10	11th Grade	Polynomial functions, theory of equations.
11	12th Grade	Circular functions, trigonometry.
12	12th Grade	Postulational systems.
13	12th Grade	Analytic Geometry.

The discussion that followed centered around the question of finding a middle ground between a whole deductive system and none, "rigor" versus "intuitively obvious."

Begle: Report from the School Mathematics Study Group. The aim of the School Mathematics Study Group (SMSG) is to improve the amount and quality of mathematical

training in the secondary and elementary schools. It is attempting to make mathematics courses in the schools more interesting so that more students are attracted, to improve the curriculum by working for better mathematics, and to help teachers prepare themselves to instruct these improved courses. SMSG is financed by the National Science Foundation. It hopes to work toward its goals by a joint effort of representatives from all parts of the mathematical profession.

There are three projects in progress. The first is the production of sample textbooks for grades 9 through 12. The second is the production and testing of several experimental units of instruction for the 7th and 8th grades; these units are being tested in a large number of classrooms in all parts of the country. Both these projects will involve writing sessions in the summer of 1959. The third project under way is the production of a series of monographs designed for the better students in high schools. These monographs are intended to supplement the high school program by showing something of the scope and interest of mathematics in our culture.

There are two projects being organized, one on films and television as teaching aids, and another on the production of teacher training materials. In addition there are various projects under consideration, such as a study of the mathematics of elementary schools, and a study of such topics as concept formation and attitudes towards mathematics. This latter study would enlist the aid of psychologists and other social scientists.

Kemeny: Courses for Teacher Training. The speaker outlined a proposed mathematics requirement for high school mathematics teacher training programs that consisted of 36 semester hours of undergraduate mathematics courses. During the first two years the student would carry one course each semester and complete the work that is roughly equivalent to Universal Mathematics I and II, and a course in the calculus of $y=f(x)$. During the junior and senior years the student would carry two courses each semester in order to complete four one-year units specially designed for teachers. These four units would consist of:

(a) One unit in modern algebra: half of it in groups, *etc.*; and half in linear algebra.
(b) One unit in geometry: substantial work in analytic geometry with introductions to projective and non-Euclidean geometrics.
(c) One unit in probability and statistics: half of it a course in probability theory, half of it a course in statistical inference.
(d) One unit in the history of mathematics: half of it to cover the period from Euclid to non-Euclidean geometry, half to cover the last 120 years, with special emphasis on topics not covered in other courses.

The speaker went on record as being in favor of a statement by the MAA concerning minimum requirements for training of high school mathematics teachers.

The discussion that followed centered about three questions:

1. Are the standards set forth by the speaker realistic?
2. Can we give students better training in mathematics and expect them to remain in the high school teaching profession?
3. Should we not be more vitally concerned with the teachers in the high school teacher training programs?

Zemansky: The Relation of Mathematics to Physics Instruction. The speaker felt that it would be of considerable value to get a short article into one of the physics journals on the trends in mathematical instruction which had been presented by previous speakers. He urged that calculus be taught early in the college program so that students would have the subject available for use in their study of physics. The speaker questioned the emphasis on understanding and insight, not for the physics students, but for the pre-engineering students, on the grounds that there was a great deal of manipulative technique that the students needed.

Discussion: Several persons present responded to this last point of the speaker by giving evidence of the need for understanding in mathematics by engineering students.

Madow: The Mathematical Training of Social Scientists. The speaker reported an increased interest among social scientists concerning the mathematical training of their students, but that many departments in the social science areas could increase the use of mathematics in their own courses. It was pointed out that the ordinary use of mathematics in the social sciences is to construct theories rather than to work out problems. The speaker also noted that the major use of mathematics is the use of probability and statistics and hence it is not so important that the average social science student get his calculus early. Since it is important that physical scientists get calculus early, the speaker asked that the CUP reconsider the proposal of the original CUP concerning a common first year mathematics course for all students.

Hoel: Undergraduate Statistics in a Mathematics Department. The speaker addressed his remarks to those situations where a college or university has neither a department of statistics nor a collection of statisticians in a mathematics department. First, an elementary service course in statistics should be given in a mathematics department only if the department has someone trained in statistics and the department is much better equipped than other departments to give it. A year course in statistics without mathematical prerequisites is more useful than a one semester course requiring a semester of mathematics first.

Next, in the case of a statistics course with a calculus prerequisite, a knowledge of integration should be insisted upon since it is inefficient to teach such a course with lower prerequisites. Integral calculus is the major tool in beginning statistical theory.

The speaker also commented on universal mathematics courses, and urged that statistics be not included as a topic in such programs. The reasons for this are that it would be a duplication for those who will be taking a statistics course, and for the others the time available is inadequate to do justice to a good explanation of statistical theory.

Regarding the undergraduate program in algebra in its relation to statistics, the speaker urged the replacement of the traditional theory of equations course by an elementary course in linear algebra with a strong orientation towards geometry.

Forsythe: The Role of Numerical Analysis in an Undergraduate Program. The speaker pointed out that most of our mathematics students major in other fields and are primarily interested in the applications of mathematics, and not in its structure. He also pointed out that about half of those who attain a Ph.D. in mathematics go into industry and apply mathematics in various areas, and that almost all of them will be connected in some degree with automatic computation. The speaker noted that there is now a strong demand for the A.B. mathematician, but that his training has not made him a good practitioner of mathematics. The speaker suggested that the aims of an undergraduate mathematics education are the following:

a. To learn as much as possible about the structure of mathematics.
b. To learn to read independently the mathematical literature at his level.
c. To know the tools of mathematics, books and machines, and how and where to find them.
d. To cultivate and practice solution of mathematical problems new to him.
e. To go fairly deeply into some other field of knowledge where mathematics is used.
f. To learn to enjoy mathematical study.

The speaker criticized much of mathematical education as not intuitive enough nor well enough illustrated, and stated his agreement with the words of Felix Klein who said, "The living thing in mathematics, its most important stimulus, its effectiveness in all directions, depends entirely upon the applications."

The speaker stated his belief that an increased role must be given to numerical

analysis in the undergraduate program, and that for the most part this work should be mixed into undergraduate courses rather than to have a great many separate courses in numerical analysis. The speaker also favored a special coding course for all students, to be taken early in their studies.

The discussion that followed brought out the need to distinguish carefully between a vocational point of view and the point of view of illustrating the fundamental ideas in mathematics by computing techniques. It was noted that even a coding course gives training in precise thought since it forces a student to think in terms of exactly what he is doing, since he must write a complete set of directions.

Cohen: Report on Films and Television for Mathematical Instruction. The speaker pointed out that films and television as a means of mathematical instruction have some, but not all, of the properties of books and teachers; consequently their use is not to be undertaken with the object of replacing either of the older aids to learning. At the October 18 meeting of the MAA's Committee on Production of Films, it was decided

1. To produce motion picture films of three hour lectures.
2. To aim the instruction at highly competent students.
3. To make one film spanning the competence of high school seniors and college freshmen.

The topics and level under consideration are:

1. Mathematical Induction (H. S. Senior—College Freshmen)
2. Theory of Limits or Integration (Sophomore or Junior Undergraduates)
3. Topology (Undergraduate Math. Club)

Correspondence is under way to determine a lecturer for each topic. Questions relating to distribution, royalties and property rights in the films to be produced were recognized as requiring answers. These answers will be sought in due course. The cost of the program will be met through a grant which the Association has obtained from the National Science Foundation.

1. The preliminary script should be a draft of the lecture prepared by the lecturer.
2. The director and lecturer then collaborate in preparation of the script based on the draft lecture.
3. The director, lecturer and committee meet to review the script and production before the making of the film.
4. The committee should be present at the making of the first film.
5. There should be a mathematician present during the making of each film to catch slips overlooked by the lecturer and director.
6. Short, inexpensive tests for image and voice should be made as an aid to the choice of a lecturer.
7. Animation may be introduced into the films as an aid to clarity in blackboard writing and diagramming.

In the long discussions that followed the report many representatives reported experimental use of television and of the large lecture system. There seemed to be general agreement that there is student acceptance of the large lecture system, and it is preferred at this time to pure television courses. A question was raised regarding the merits of such instruction for the gifted student. It was generally agreed that special instruction should be provided for the "top 1%," but there was not general agreement as to whether or not the new CUP should concern itself with such special instructional programs.

Begle: Discussion of writing sessions. The speaker described the different kinds of writing programs which had been operated: small writing groups as contrasted with large ones, and both of these compared with scattered efforts by individuals.

Kemeny: Financial Arrangements for Authors. The speaker cited the experience of the Dartmouth writing group, which did not receive royalties from its published works. It was felt that in the future such limitations should be removed.

Discussion: There was a considerable response to this matter of financial arrangements. The basic problem is the granting of royalty rights to writers who have been paid for their writing efforts. It was felt that if an author was paid to write a book and in addition received full royalties, there might be criticism from the free-lance mathematical writers. This criticism could be met by subtracting the initial payment to the writer from subsequent royalty payments. Many preferred to avoid any such subtraction, in order to provide adequate economic motivation for high calibre persons in writing projects.

Discussion: The Nature of Books to be Written. The question discussed was the desirability of source-books vs. textbooks. Since source-books are written too compactly for classroom use, their role is to serve as materials for use by writers of texts. The experience of some writers of source-books is to say "never again!," but it was urged that some compromise stage between source-books and textbooks be sought.

RESOLUTIONS ADOPTED BY THE CUP CONFERENCE

Since the Committee on the Undergraduate Program is charged with responsibility for advising the Mathematical Association of America on all matters connected with the undergraduate program in mathematics, this Conference hereby adopts the following recommendations concerning certain special aspects of the work of the Committee:

1. This Conference recommends that the Committee on the Undergraduate Program in Mathematics consist of from ten to fifteen persons and that the Committee have power to delegate its various activities to subcommittees whose members need not be members of the Committee.

2. This Conference recommends that the Committee on the Undergraduate Program be provided with adequate facilities and staff, free of other responsibilities, sufficient to carry out the tasks assigned to it.

3. This Conference recommends that the Committee on the Undergraduate Program continue to be concerned with the development of undergraduate courses in mathematics and with the production of adequate materials and methods for these courses, keeping in mind the varying needs of students destined for the many careers which make use of mathematics, and particularly the needs of prospective high school mathematics teachers.

4. This Conference recommends that the Committee on the Undergraduate Program continue to plan its activities with due regard for other groups concerned with similar problems.

5. Since the encouragement of writing of adequate mathematical textbooks is a primary problem facing the Committee on the Undergraduate Program, this Conference recommends to the Board of Governors of the Association and to the Committee that the Committee develop a program to subsidize the preparation of textbooks and monographs without prejudice to the royalty rights of the authors.

6. This Conference recommends that, upon the recommendation of the Committee on the Undergraduate Program, the Mathematical Association of America publish a statement of minimal standards for teachers of mathematics in high schools, junior colleges, and colleges. The review and revision of such standards should be a continuing responsibility of the Committee and the Committee should recommend to the Association future changes in such standards. The existence of such statements of minimal

standards should be given wide publicity through the Sections of the Association to all interested state and local groups.

7. The Conference recommends that the Committee on the Undergraduate Program make recommendations to the Association regarding the desirable mathematical preparation of students expecting to pursue graduate work in mathematics or expecting to pursue mathematical careers in industry.

8. Since the vastly increased demands upon mathematicians and upon teachers of mathematics in the undergraduate colleges of this country have put upon the Mathematical Association of America the obligation of providing professional leadership in the design of appropriate courses to satisfy these demands, and since the obligations of the Association in this field have been delegated to the Committee on the Undergraduate Program, and since this delegation implies a vastly expanded program of activity as described in these resolutions, therefore this Conference recommends that the officers and the Board of Governors of the Association communicate to the appropriate agencies the compelling need for grants of funds commensurate with the magnitude of the task and with its importance for the development of mathematics and science in this country.

Concluding note. A Committee on the Undergraduate Program in Mathematics is now being appointed. Members of this committee met in New York City on December 29 and 30, 1958 to plan its future activities in view of the recommendations of the Conference.

R. C. FISHER AND IVAN NIVEN, *Recorders*
HARRY M. GEHMAN, *Secretary-Treasurer*

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

NOTE ON HYPERGEOMETRIC POLYNOMIALS

S. K. CHATTERJEA, Bangabasi College, Calcutta, India

1. In a recent paper [1], V. R. Thiruvenkatachar and T. S. Nanjundiah have proved the following nonlinear recurrence relation for ultraspherical polynomials:

$$(1.1)\quad (1-x^2)D_n^{\lambda}(x) = n(n+2\lambda)[P_n^{\lambda}(x)]^2 - (n+1)(n+2\lambda-1)P_{n-1}^{\lambda}(x)P_{n+1}^{\lambda}(x),$$

where

$$D_n^{\lambda}(x) \equiv [DP_n^{\lambda}(x)]^2 - [DP_{n-1}^{\lambda}(x)][DP_{n+1}^{\lambda}(x)].$$

M. S. Webster [2] has pointed out that (1.1) characterizes the ultraspherical polynomials. We show in this section that a very simple proof of the formula (1.1) may be given merely by considering the following two differentiation formulas ([3], vol. 2, p. 176):

$$xDP_n^{\lambda}(x) = DP_{n-1}^{\lambda}(x) = nP_n^{\lambda}(x), \tag{1.2}$$

$$DP_{n+1}^{\lambda}(x) - xDP_n^{\lambda}(x) = (n + 2\lambda)P_n^{\lambda}(x). \tag{1.3}$$

Changing n into $n+1$ in (1.2) and $n+1$ into n in (1.3) we obtain

$$xDP_{n+1}^{\lambda}(x) - DP_n^{\lambda}(x) = (n + 1)P_{n+1}^{\lambda}(x) \tag{1.4}$$

and

$$DP_n^{\lambda}(x) - xDP_{n-1}^{\lambda}(x) = (n + 2\lambda - 1)P_{n-1}^{\lambda}(x). \tag{1.5}$$

Next from (1.2) and (1.3) we have on multiplication

$$-x^2(DP_n^{\lambda})^2 - DP_{n-1}^{\lambda} \cdot DP_{n+1}^{\lambda} + x[DP_n^{\lambda}DP_{n+1}^{\lambda} + DP_{n-1}^{\lambda} \cdot DP_n^{\lambda}] = n(n + 2\lambda) \cdot (P_n^{\lambda})^2. \tag{1.6}$$

Similarly from (1.4) and (1.5) we get

$$(DP_n^{\lambda})^2 + x^2DP_{n-1}^{\lambda} \cdot DP_{n+1}^{\lambda} - x[DP_n^{\lambda} \cdot DP_{n+1}^{\lambda} + DP_{n-1}^{\lambda} \cdot DP_n^{\lambda}] = -(n + 1)(n + 2\lambda - 1)P_{n-1}^{\lambda} \cdot P_{n+1}^{\lambda}. \tag{1.7}$$

Finally from (1.6) and (1.7) we get on addition

$$(1 - x^2) \cdot [(DP_n^{\lambda})^2 - (DP_{n-1}^{\lambda}) \cdot (DP_{n+1}^{\lambda})] = n(n + 2\lambda)(P_n^{\lambda})^2 - (n + 1)(n + 2\lambda - 1)P_{n-1}^{\lambda}P_{n+1}^{\lambda}.$$

We would like to point out that we succeeded in extending this result (1.1) as well as the characterization [4] by using the Jacobi polynomials in place of the ultraspherical polynomials.

2. The following formula is due to the German mathematician Alexander Dinghas [5]: If $P(x)$ denotes a linear polynomial in x of the form $Ax+B$, which remains positive throughout the closed interval $[a, b]$ and if α, β are two real positive numbers, we have

$$\frac{1}{P^{\alpha}(a)P^{\beta}(b)} = \frac{(b - a)^{1-\alpha-\beta}}{B(\alpha, \beta)} \int_a^b \frac{(b - x)^{\alpha-1}(x - a)^{\beta-1}}{P^{\alpha+\beta}(x)}\, dx, \tag{2.1}$$

where $P^{\alpha}(a) = (Aa+B)^{\alpha}$ and $B(\alpha, \beta)$ is Euler's integral of the first kind. We wish to give in this section several interesting results, *viz.*, (2.2), (2.5), (2.6), (2.7), (2.8), (2.9), all derived from (2.1).

Substituting $x=a\sin^2\theta+b\cos^2\theta$, we at once write (2.1) as follows:

$$\frac{1}{P^{\alpha}(a)P^{\beta}(b)}=\frac{2}{B(\alpha,\beta)}\int_0^{\pi/2}\frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{\{P(a\sin^2\theta+b\cos^2\theta)\}^{\alpha+\beta}}d\theta. \tag{2.2}$$

From (2.2) it is easy to verify the well-known definite integral ([3], vol. I, p. 11)

$$\int_0^{\pi/2}\frac{(\sin\theta)^{2\alpha-1}\cdot(\cos\theta)^{2\beta-1}}{(a\sin^2\theta+\cos^2\theta)^{\alpha+\beta}}d\theta=\frac{1}{2}a^{-\alpha}B(\alpha,\beta),\quad 0<a<1. \tag{2.3}$$

Next putting $P(x)=1-2xh+h^2$ in (2.2) we get for small h

$$(1-2ah+h^2)^{-\alpha}(1-2bh+h^2)^{-\beta}=\frac{2}{B(\alpha,\beta)}\int_0^{\pi/2}\frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{\{1-2(a\sin^2\theta+b\cos^2\theta)h+h^2\}^{\alpha+\beta}}d\theta. \tag{2.4}$$

Now we remember as usual the ultraspherical polynomial $P_n^{\lambda}(x)$, $(\lambda\neq 0)$, by the expansion

$$(1-2hx+h^2)^{-\lambda}=\sum_{n=0}^{\infty}P_n^{\lambda}(x)h^n.$$

Thus equating coefficients of h^n from both members of (2.4) we get

$$\sum_{r=0}^{n}P_r^{\alpha}(a)P_{n-r}^{\beta}(b)=\frac{2}{B(\alpha,\beta)}\int_0^{\pi/2}P_r^{\alpha+\beta}(a\sin^2\theta+b\cos^2\theta)(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}d\theta. \tag{2.5}$$

In particular when $\alpha=\beta=\frac{1}{2}$ we get from (2.5)

$$\sum_{r=0}^{n}P_r(a)P_{n-r}(b)=\frac{2}{\pi}\int_0^{\pi/2}U_n(a\sin^2\theta+b\cos^2\theta)d\theta, \tag{2.6}$$

where $P_n(x)$ and $U_n(x)$ denote as usual the Legendre and Chebyshev (second kind) polynomials respectively.

Again putting $\alpha=\frac{1}{2}$, $\beta=1$ in (2.5) we obtain a more interesting result

$$\sum_{r=0}^{n}P_r(a)U_{n-r}(b)=4\cdot\int_0^{\pi/2}P'_{n+1}(a\sin^2\theta+b\cos^2\theta)\cos\theta d\theta, \tag{2.7}$$

where $P'_n(x)\equiv(d/dx)\{P_n(x)\}$.

Or we can write (2.7) as follows:

$$\sum_{r=0}^{n}U_r(a)P_{n-r}(b)=4\cdot\int_0^{\pi/2}P'_{n+1}(a\sin^2\theta+b\cos^2\theta)\sin\theta\,d\theta. \tag{2.8}$$

Lastly putting $a=-b$ in (2.6) and then changing b into x we get

$$(2.9)\qquad \sum_{r=0}^{n}(-1)^r P_r(x)P_{n-r}(x) = \frac{2}{\pi}\int_0^{\pi/2} U_n(x\cos 2\theta)d\theta.$$

I am indebted to Dr. H. M. Sengupta (Department of Pure Mathematics, Calcutta University) for his kind help in the preparation of this note.

References

1. V. R. Thiruvenkatachar and T. S. Nanjundiah, Inequalities concerning Bessel functions and orthogonal polynomials, Proc. Indian Acad. Sci., vol. 33, 1951, pp. 373–384.

2. M. S. Webster, Non-linear recurrence relations for certain classical functions, this MONTHLY, vol. 64, 1957, pp. 249–252.

3. E. H. Bateman, Higher Transcendental Functions, New York, 1953.

4. S. K. Chatterjea, Notes on Jacobi and related polynomials, Bull. Calcutta Math. Soc., vol. 50, 1958, pp. 77–83.

5. Alexander Dinghas, Zur Darstellung einiger Klassen hypergeometrischer Polynome durch Integrale vom Dirichlet-Mehlerschen Typus, Math. Z., vol. 53, 1950, pp. 76–83.

AN INEQUALITY FOR THE FOURIER COEFFICIENTS OF A NONNEGATIVE FUNCTION

J. D. WESTON, King's College, University of Durham, England

In the analysis of an electrical network consisting of linear elements combined with a periodically varying conductance, one is led to represent the conductance function $g(\theta)$ by its Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

where, for each n,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in\theta}d\theta.$$

Under certain assumptions, it can be shown that the system responds linearly to small perturbations, and is stable provided that

$$(1)\qquad c_0^2 + c_0|c_{2n}| \geqq 2|c_n|^2.$$

On the other hand, energy considerations suggest the system will be stable if $g(\theta)$ is never negative. One is thus invited to prove that (1) holds for any nonnegative function $g(\theta)$ which has Fourier coefficients c_n. This can be done in the following way.

For a given value of n, we can choose a real number α so that $e^{in\alpha}c_n$ is real. Then

$$e^{in\alpha}c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in(\theta-\alpha)}d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\cos n(\theta-\alpha)d\theta,$$

since $g(\theta)$ is real. Also,

$$|c_{2n}| \geqq \Re(e^{2in\alpha}c_{2n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\cos 2n(\theta-\alpha)d\theta.$$

Now suppose that $g(\theta)\geqq 0$ for all θ. Then $c_0\geqq 0$, so that (1) is established if we can show that

$$c_0^2 + c_0\Re(e^{2in\alpha}c_{2n}) - 2|c_n|^2 \geqq 0. \tag{2}$$

Denoting the left-hand side of (2) by L, we have

$$4\pi^2 L = \left\{\int_{-\pi}^{\pi} g(\theta)d\theta\right\}^2 + \int_{-\pi}^{\pi} g(\theta)d\theta\int_{-\pi}^{\pi} g(\theta)\cos 2n(\theta-\alpha)d\theta$$

$$- 2\left\{\int_{-\pi}^{\pi} g(\theta)\cos n(\theta-\alpha)d\theta\right\}^2$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(\theta)g(\phi)\{1+\cos 2n(\phi-\alpha) - 2\cos n(\theta-\alpha)\cos n(\phi-\alpha)\}d\theta d\phi.$$

Thus

$$2\pi^2 L = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(\theta)g(\phi)\cos n(\phi-\alpha)\{\cos n(\phi-\alpha) - \cos n(\theta-\alpha)\}d\theta d\phi.$$

We may interchange θ and ϕ, so that

$$2\pi^2 L = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(\theta)g(\phi)\cos n(\theta-\alpha)\{\cos n(\theta-\alpha) - \cos n(\phi-\alpha)\}d\theta d\phi.$$

Adding these two expressions for $2\pi^2 L$, we see that

$$4\pi^2 L = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(\theta)g(\phi)\{\cos n(\theta-\alpha) - \cos n(\phi-\alpha)\}^2 d\theta d\phi.$$

It follows that $L\geqq 0$, equality occurring only if $g(\theta)\equiv 0$.

The inequality (1) is fairly delicate, for although $c_0\geqq|c_n|$ if $g(\theta)\geqq 0$ for all θ, we can also have $c_0+|c_{2n}|<2|c_n|$: for example, if $g(\theta)=1$ when $\cos\theta\geqq 0$ and $g(\theta)=0$ when $\cos\theta<0$, then $c_0=\frac{1}{2}$, $c_1=1/\pi$, and $c_2=0$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

LAGRANGE'S MULTIPLIERS

C. SALTZER, Case Institute of Technology

In most expositions of the Lagrange multiplier method, no motivation for the introduction of the multipliers is given. The discussion below presents a geometric interpretation.

A stationary point of a function $\phi(x_1 \cdots x_n)$ is, by definition, a point at which the gradient $(\phi_1, \cdots, \phi_n)$, where $\phi_k = \partial\phi/\partial x_k$ $(k = 1, \cdots, n)$, is zero. If we regard the vector $(dx_1 \cdots dx_n)$ as a permissible displacement the condition that the gradient be zero may be replaced by the condition that the gradient be orthogonal to all permissible displacements which, in the unconstrained case, is any arbitrary displacement. If the function is subjected to m independent constraints

$$f_k(x_1, \cdots, x_n) = 0 \qquad (k = 1, \cdots, m),$$

then the permissible displacements are precisely those which are orthogonal to the gradients of these functions since

$$df_k = f_{k1}dx_1 + \cdots + f_{kn}dx_n = 0 \qquad (k = 1, \cdots, m),$$

where $f_{kr} = \partial f_k/\partial x_r$. The condition that the gradient of the function, ϕ, be orthogonal to all permissible displacements is satisfied if and only if the gradient of ϕ lies in the space spanned by the gradients of the constraining functions, *i.e.*, the gradient of ϕ must be a linear combination of the gradients of the constraining functions. Thus, if a new function is formed by adding a linear combination of the constraining functions to ϕ, and the gradient of the new function is equated to zero, the last condition is satisfied.

The above discussion is applicable also to nonholonomic constraints for which the gradient of the constraints is replaced by the vector whose components are the coefficients of the differentials in the form which defines the constraint.

A NOTE ON THE CONICS

J. GALLEGO-DIAZ, University of Puerto Rico

The purpose of this paper is to study one application of the transformation $\Omega = z^2$, where Ω and z are complex. We consider a given central conic in the Ω-plane with a focus at the pole and its major axis coinciding with the polar axis.*

* I have not seen such an application studied in any of the books that I have consulted, nor in the excellent *Dictionary of Conformal Representations*, by H. Kober, New York, 1952.

We suppose that the points of the Ω-plane are given either by cartesian coordinates (u, v) or by polar coordinates (ρ_1, ω_1). Similarly, points in the z-plane are given either by cartesian coordinates (x, y) or by polar coordinates (ρ, ω).

The polar-coordinate equation of a conic with a focus at $(0, 0)$ and its major axis on the polar axis is

$$\rho_1 = \frac{p_1}{1 - e_1 \cos \omega_1}. \tag{1}$$

The transformation we are considering may be written either as

$$u = x^2 - y^2, \qquad v = 2xy \tag{I}$$

or as

$$\rho_1 = \rho^2, \qquad \omega_1 = 2\omega. \tag{II}$$

By virtue of these, (1) becomes

$$\rho^2 = \frac{p}{1 - e_1 \cos 2\omega} \tag{2}$$

or

$$\frac{x^2}{p_1/(1 - e_1)} + \frac{y^2}{p_1/(1 + e_1)} = 1, \tag{3}$$

which means that the transformed curve in the z-plane is another conic whose center is the origin and whose axes coincide with the coordinate axes.

Letting a and b denote the semiaxes of the conic (3) and a_1 and b_1, those of the conic (1), we easily obtain

$$b^2 = a_1 - c_1, \qquad a^2 = a_1 + c_1; \tag{4}$$

$$a^2 + b^2 = 2a_1, \qquad ab = \pm b_1; \tag{5}$$

$$c_1 = c^2/2, \qquad e_1 = e^2/(2 - e^2), \tag{6}$$

where e_1 and e are the respective eccentricities.

It is seen without difficulty that if $e_1 < 1$ then $e < 1$ and that if $e_1 > 1$ then $e > 1$. In other words, *the transformation preserves the nature of the conic.*

Keeping in mind formulas (4), (5), and (6), we are able to prove easily and quickly a large number of theorems and to solve numerous problems concerning loci, envelopes, maxima and minima, orthogonal trajectories, graphical constructions, and so on. In some cases it is enough to remember that a straight line in the Ω-plane is transformed into a rectangular hyperbola with center at the origin in the z-plane and that a straight line in the z-plane is transformed into a parabola with focus at the origin in the Ω-plane. These results follow from (I) and (II) since $Mu + Nv + P = 0 \rightarrow M(x^2 - y^2) + 2Nxy + P = 0$ and

$$\frac{1}{\rho} = A\cos\omega + B\sin\omega \rightarrow \frac{1}{\sqrt{\rho_1}} = A\cos\tfrac{1}{2}\omega_1 + B\sin\tfrac{1}{2}\omega_1$$

$$\rightarrow \frac{2}{\rho_1} = A^2 + B^2 + (A^2 - B^2)\cos\omega_1 + 2AB\sin\omega_1.$$

The transformation is conformal and its critical points are $z=0$ and $z=\infty$.

Other important and obvious properties of this transformation are:

(a) To diametrically opposite points on the conic in the Ω-plane correspond the ends of semiconjugate diameters of the transformed conic in the z-plane. ($P\rightarrow M$, $Q\rightarrow L$ in Fig. 1.)

(b) The two foci F and F' of the conic in the z-plane correspond to the other focus O' of the conic in the Ω-plane (Fig. 1).

(c) The ends of two orthogonal semidiameters of the conic in the z-plane correspond to the ends of a focal chord of the conic in the Ω-plane.

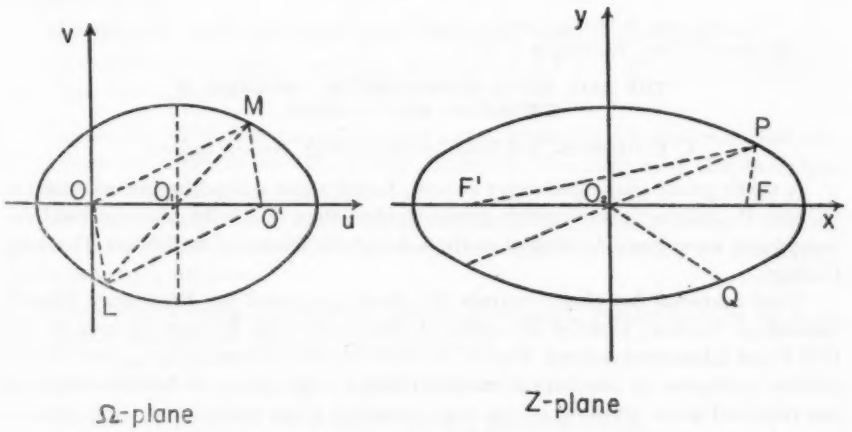


FIG. 1

We give two examples to illustrate simple proofs.

Example 1. Let O_2P and O_2Q be two semiconjugate diameters of the given ellipse in the z-plane, whose foci are F and F' corresponding to the focus O' of the given ellipse in the Ω-plane (Fig. 1). Since $MO'=OL$, it follows that $PF\cdot PF'=\overline{OQ}^2$. In other words, *the product of the radius vectors of an ellipse which begin at the point P is equal to the square of the semiconjugate diameter of* O_2P.

Example 2. Since $OM+OL=2a_1$, it follows that $\overline{O_2P}^2+\overline{O_2Q}^2=a^2+b^2$. (Theorem of Appolonius.)

The following are further examples of results that may easily be obtained:

1. The envelope of a family of coaxial ellipses which have equal area consists of two rectangular hyperbolas.

2. Let H_1 and H_2 be two rectangular hyperbolas whose common center is A and which are tangent at M and N, respectively, to a given ellipse whose center is also A. If H_1 and H_2 intersect at P, prove that AP bisects the angle MAN.

3. Let OA and OB be two orthogonal semidiameters of a given ellipse. Prove that $1/\overline{OA}^2+1/\overline{OB}^2=1/a^2+1/b^2$.

4. The orthogonal trajectories of the family of curves $(x^2+y^2)^2=k^2(x^2-y^2)$ are the family of curves $(x^2+y^2)^2=2c^2xy$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware AND JOHN R. MAYOR, AAAS and University of Maryland

Contributions for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

THE BALL STATE EXPERIMENTAL PROGRAM IN GEOMETRY AND ALGEBRA

C. F. BRUMFIEL, Ball State Teachers College, Muncie, Indiana

A tenth grade plane geometry course, based upon a modified version of the Hilbert Postulates, and a ninth grade algebra that is a mild postulational development have been developed in the laboratory school of Ball State Teachers College.

Text material for these courses has been prepared by Professors Merrill Shanks of Purdue, Charles Brumfiel of Ball State and Robert Eicholz of the Ball State laboratory school. Under National Science Foundation grants for in-service institutes for teachers of mathematics a large group of Indiana teachers has received some training in the mathematical ideas underlying these experimental programs. The geometry is being taught experimentally in seven schools and the algebra in six. A classroom visitation and supervisory program is in effect for the teachers of this material.

Both the geometry and algebra are clear-cut attempts to introduce students to the axiomatic structure of mathematics. The geometry is at a high level of rigor. The use of axiomatics in algebra, however, is primarily for clarity rather than for proof. The point of view taken is that beginning algebra should be primarily a study of the algebra of the rational number system with an introduction to real numbers. This is the fourth year for the geometry. It has been through three revisions and is essentially in final form. A complete teachers' manual has been prepared. The algebra is a second edition and will undergo considerable revision during the next year.

In the early part of the geometry heavy emphasis is placed upon the fact that mathematical lines, points and other geometric concepts exist only as ideas, not as physical entities. Students enjoy viewing a chalk mark upon the board as a "batch of chalk dust" hanging there. They are quick to recognize that such a collection of chalk molecules has many properties that we do not wish to ascribe to lines.

Eighteen postulates are formulated. Fifteen of these are postulates of *incidence*, *betweenness*, and *congruence*. Archimedes' postulate, the completeness postulate, and the parallel postulate complete the axiom set. There is no attempt to employ a minimal set of assumptions. Two major modifications of the Hilbert axioms have been made. Pasch's axiom is replaced by the postulate that every line separates the plane. Now Pasch's axiom (a line not on any vertex of a triangle and cutting one side cuts a second side) is an easy theorem. The second modification is in the formulation of the completeness postulate. Before this postulate is introduced *length of a segment* has been *defined*, using betweenness, congruence, and the axiom of Archimedes. The completeness postulate is stated as below.

For every line l and every point A on l and every positive real number x there is, on a given side of A, a point B such that the length of AB is x. (Of course a unit segment has been chosen.)

Here is the only place in the geometry that actual cheating occurs. The student does not really know what a real number is. However, with this exception the text is believed to be rigorous.

Having available the betweenness postulates (Euclid's chief omission), which are not available to authors of conventional texts, it is possible to define precisely concepts such as:

Segment, ray, angle, interior and exterior of an angle, greater than for segments and angles, addition of segments and angles, etc.

Some illustrative definitions are listed below.

Every set of points in the plane is called a geometric figure.

A set of points consisting of two points and all points between them is called a line segment.

The interior of angle AOB is the set of all points that are both on A's side of line OB and also on B's side of line OA.

Another distinctive feature of the Ball State geometry is a unit on logic done in the spirit of such texts as Allendoerfer and Oakley, *Principles of Mathematics* and Kemeny, Snell and Thompson, *Introduction to Finite Mathematics*.

The comments above stress the differences between the Ball State geometry and conventional texts. The similarities should also be mentioned. Roughly, the first third of the year is spent upon logic, betweenness, and congruence.

After this introduction the usual theorems of plane geometry are considered and proofs are much the same as in conventional classes. Experience has shown that students who transfer out of the Ball State geometry to other schools during the school year can adjust without difficulty to conventional courses. However, it is not easy for students with a conventional geometry background to step into the Ball State geometry course after the first few weeks. The geometry engenders a considerable amount of enthusiasm among good students and teachers. A testing program last year which compared 250 students in the experimental classes to about the same number of students in conventional classes produced results favorable to the experimental group.

Throughout the algebra the axiomatic structure is held somewhat in the background. Much is postulated that is demonstrable. In order to emphasize the structure of the algebra of the rational number system, the postulates for the set consisting of zero and the natural numbers (existence and uniqueness, commutativity, associativity, distributivity, cancellation laws, and properties of 0 and 1) are repeated for the set of integers and then reiterated for the set of rational numbers. Students are encouraged to proceed intuitively and are rarely expected to present formal proofs. However, the teacher guides the class through several correct proofs.

Some systematic study of logic is made. Concepts of set theory are utilized and the language of quantification is used. In early work equations and inequalities are treated together. In many examples the variables are restricted to finite sets for simplicity.

When the negatives of the natural numbers are introduced it is at once postulated that the old familiar laws for the counting numbers hold for these new numbers. Now the usual calculation rules, *e.g.*, $a(-b) = -(ab)$, are proved as theorems.

The conventional topics of elementary algebra, linear equations, work with formulas, computation with polynomials, factoring, graphing, ratio and proportion, fractional equations, *etc.* are treated. But, emphasis is always placed upon the kinds of numbers that are being used. For example, computations that involve fractional equations are treated as computations with rational numbers.

In the Ball State geometry classes this year are some students who have had one year of the experimental algebra. These students adjust more easily to the abstract concepts of geometry than do the students who have come through traditional algebra classes.

A GEOMETRY COURSE FOR JUNIORS

CHARLES C. BUCK, University of Alabama

The geometry course described below is intended to be a course which grows naturally out of the student's background in high school geometry. It has been taught for several years to upperclassmen at the University of Alabama. Most of the students have been mathematics majors in the College of

Arts and Sciences or in the College of Education, but qualified liberal arts students, such as majors in philosophy, are welcome. A variant of the course is regularly taught in the summer school to high school geometry teachers working on advanced degrees.

The student at the post-calculus level needs some mathematics courses which bring him back to the realization that mathematics is deductive argument. In this connection an important committee of mathematicians has said, "high school geometry is usually the only course in high school or the early years of college which is taught as a mathematical subject" [1]. It is true that there is now an increasing emphasis on deduction in the elementary courses and that computation is itself a form of deduction, but in the mind of the student this feature is obscured by the need for knowing that $D_x \ln x = 1/x$ and that $\sin 2A = 2 \sin A \cos A$. The mathematics majors, who may have hungered for logical argument through years in which the emphasis was on the formal manipulation of symbols, deserve to be told that mathematics is proof and that logical proof is a vehicle in which new territory can be discovered and explored. It is a major purpose of this course to demonstrate this.

This course starts with a study of Book One of Euclid's *Elements* [2]. We note the obvious flaws and indicate how these may be patched up. (In general, I take a positive attitude and encourage the students to pick the flaws.) We also note the branch points into the two major non-Euclidean geometries. (The clean separation that Euclid made between those theorems dependent on his parallel postulate and those which are not, led Coxeter to refer to him [3] as "the first non-Euclidean geometer.")

Having used the *Elements* to establish a frame of reference (and as a review of the relevant parts of high school geometry), we explore a little way down the path opened up by Bolyai and Lobatschewsky—far enough to cover one topic in some detail. For the foundations of hyperbolic geometry, we read the first part of Lobatschewsky's *Theory of Parallels* (bound in [4]). For the past year or so we have studied, as the additional topic, the theory of plane area, presented by means of lectures based on the article *Area in non-Euclidean geometry* [5] by Kenneth Leisenring. In other years we have studied the circles and spheres of hyperbolic space and proved the theorem that the geometry on a horosphere is Euclidean.

The point at which the path to elliptic geometry branches off from Euclid is somewhat obscured, but the students see it easily when it is pointed out to them. We explore elliptic geometry at least far enough to see that every triangle has an angular excess and that there is a natural unit of length: the polar distance. If we have studied area in the hyperbolic plane, I ask for papers on area in the elliptic plane. If we have studied circles and spheres in hyperbolic geometry (which is a larger topic than area), there is no corresponding topic of equal interest and accessibility in elliptic geometry.

By the time we have built this much of a superstructure on our patched up

Euclid, the student has quite a stake in the validity of his reasoning processes. It may therefore come as a surprise to him that the propositions "All triangles are isosceles," and "A right angle is equal to an angle which is greater than a right angle," are theorems in the system he has been using so confidently [6].

When it is realized that these paradoxes are provable because we have ignored one-dimensional geometry (linear order) and that a whole new set of axioms are needed for this, it is seen that Euclid needs more than patching up and that we must, moreover, have some assurance that no more paradoxes can crop up.

With this in mind we study the development of the axiom system in Hilbert's *Foundations of Geometry* [7]. There are theorems of various degrees of difficulty for the students to prove. In these they can get experience with the technique of rigorous proof as demanded by Hilbert's material and by their own new understanding of the meaning of proof. The way in which the flaws in Euclid are corrected in Hilbert is easily seen. With the proof of the consistency of Hilbert's axioms, the course ends.

The students are required to buy three books [2], [4], and [7] for this course. The total cost is about five dollars. The parts of these books which are not directly used in the course form a collection of valuable primary reference material.

Aside from the major purposes of orienting the student correctly in mathematics as a whole and of providing an introduction on a mature level to one branch of mathematics, this course has some worthwhile by-products. Three of these are: (1) the texts that we use are not textbooks but beautifully constructed mathematical treatises; (2) the nature of the course encourages the use of the library and helps the student to get acquainted with some of the literature of mathematics; (3) it introduces a little of the history of mathematics in a natural way and portrays the subject as continually growing and not as a static thing completed in the seventeenth century.

References

1. The 1954 Summer Writing Group of the Department of Mathematics, University of Kansas, Universal Mathematics, Part 1. Student Union Book Store, Lawrence, Kansas, p. 7. (A rearranged edition entitled *Universal Mathematics* may be obtained free of charge by writing to Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.)

2. T. L. Heath, The Thirteen Books of Euclid's Elements. 2nd ed., vol. I, Introduction and Book I, II, New York, 1956.

3. H. S. M. Coxeter, Non-Euclidean Geometry, University of Toronto Press, 1942, p. 2.

4. Roberto Bonola, Non-Euclidean Geometry with a supplement containing the Dr. George Bruce Halsted translations of The Science Absolute of Space by Johann Bolyai and The Theory of Parallels by Nicholas Lobatschewsky, New York, 1955.

5. Kenneth Leisenring, Area in non-Euclidean geometry, this MONTHLY, vol. 58, 1951, pp. 315–322.

6. W. W. Rouse Ball, Mathematical Recreations and Essays, 10th ed., 1922, London, pp. 45, 48, 49.

7. David Hilbert, Foundations of Geometry, reprint ed., LaSalle, Illinois, pp. 1–30.

CURRICULUM STUDIES IN MATHEMATICS

A report from the School Mathematics Study Group, submitted at the Washington Conference on the Committee on the Undergraduate Program, appears on pp. 215-216 of this issue.

CURRENT ITEMS

Standards for Materials and Equipment for the Improvement of Instruction in Science, Mathematics, and Modern Foreign Languages is the title of a bulletin recently issued by the Council of Chief State School Officers. The bulletin reports a conference on this subject held by the Chief State School Officers at Michigan State University in November 1958. Mathematics consultants to the chief state school officers at the conference were: John A. Brown, University of Delaware; Harry M. Gehman, University of Buffalo; Donovan Johnson, University of Minnesota. John R. Mayor of AAAS served as director of the conference. Copies of the bulletin are available through state departments of education.

A Yardstick for Measuring your Science and Mathematics Programs is the title of a 4-page brochure recently issued by the President's Committee on Scientists and Engineers. The brochure describes programs for grades 7 through 12 in science and in mathematics.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1345 [1958, 775]. *Correction.*

Change the last subscript in line 4 of the statement of the problem to $2x+1$.

E 1356. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Let I, O, r, R, K denote the incenter, circumcenter, inradius, circumradius, and area of a triangle T. Show that $(IO)^2+K=r^2+R^2$ if and only if T is a right triangle.

E 1357. *Proposed by Albert Wilansky, Lehigh University*

Prove that for every number a the equation $x=-a+\sqrt{2}\sin[(a-x)/\sqrt{2}]$ has a unique solution.

E 1358. *Proposed by N. R. Riesenberg, Brooklyn College*

Show that the roots of the cubic equation $64x^3-192x^2-60x-1=0$ are $\cos^3(2\pi/7)\sec(6\pi/7)$, $\cos^3(4\pi/7)\sec(2\pi/7)$, $\cos^3(6\pi/7)\sec(4\pi/7)$.

E 1359. *Proposed by Leo and William Moser, Universities of Alberta and Saskatchewan*

(1) Given eight positive integers $a_1<a_2<\cdots<a_8\leqq 16$. Prove that there exists a k such that $a_i-a_j=k$ has at least three solutions.

(2) Find a set $a_1, \cdots, a_8$ for which $a_i-a_j=k$ has at most three solutions for any k.

E 1360. *Proposed by J. L. Brenner, Stanford Research Institute*

Call two elements a, b of a group "k-commutative" (B. Friedman) if every product of k factors, each factor being a or b, commutes with every other such product. Show that for every two elements a, b of a group, the set of all k for which a, b are k-commutative is an ideal in the set of nonnegative integers.

SOLUTIONS

An Enumeration of Triangles

E 1326 [1958, 526]. *Proposed by P. L. Chessin, University of Maryland*

With n straight line segments of lengths 1, 2, 3, $\cdots$, n, how many non-degenerate triangles can be constructed?

Solution by C. S. Ogilvy, Hamilton College. Each time n is increased by one to an even integer, the number of new triangles added is $(n-2)^2/4$; each time to an odd integer, $(n-1)(n-3)/4$. Applying well-known summation formulas, we find that if T is the total number of triangles constructible with n segments, then

$$T = n(n-2)(2n-5)/24, \quad n \text{ even},$$

$$T = (n-1)(n-3)(2n-1)/24, \quad n \text{ odd}.$$

Also solved by Eugene Albert, P. M. Anselone, J. H. Bailey, H. F. Bennett, W. J. Blundon, Julian Braun, D. A. Breault, Robert Burton, R. W. Estus, Susan L. Friedman, Emil Grosswald, B. A. Hausmann, S.J., H. K. Hilton, A. R. Hyde, C. H. King, Morton Kupperman, W. M. McKeeman, D. C. B. Marsh, Helen M. Marston, Leo Moser, Paul Payette, N. R. Riesenberg, Ricky Ritterman, Daniel Serebrakian, and the proposer. Late solutions by Merrill Barnebey, D. R. Brillinger, Oldrich Buchta, André Dupras, S. H. Greene, Joe Lipman, Frederick Luttmann, J. B. Muskat, Walter Penney, and Benjamin Sapolsky.

Editorial Note. If a distinction is to be made between the clockwise and the counterclockwise triangles with sides a, b, c, then the answers above should be doubled. The two formulas can variously be combined into one, as for example

$$T = n(n-2)(2n-5)/24 + \{-1+(-1)^n\}/16.$$

Blundon and Kupperman employed the fact that the number of triangles with longest side m is the number of lattice points interior to the triangle with vertices $(0, m)$, (m, m), $(m/2, m/2)$.

Bailey and Braun showed that T is given by the sum of the numbers in the first $n-3$ rows of the arithmetical triangle.

```
1
1 1
1 2 1
1 2 2 1
1 2 3 2 1
1 2 3 3 2 1
1 2 3 4 3 2 1
. . . . . . . .
```

Trisection with Clocks

E 1327 [1958, 526]. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

A confirmed angle-watcher marked the angle α formed by the hands of a clock. Some time later he noticed that the hands trisected α. In how short a time could this have happened? How soon after 3:00 could one start such an experiment?

Solution by Hyman Orlin, Coast and Geodetic Survey. We have $m_0-h_0=\alpha$, where m_0 and h_0 are the angular positions (measured clockwise from the 12 o'clock position) of the minute and hour hands at time t_0. We seek a solution where the time interval Δt leading to a trisection is less than one hour. In this time interval either

$$(1) \qquad [h_0 + (2\pi/12)\Delta t] - h_0 = \alpha/3, \quad (m_0 + 2\pi\Delta t) - m_0 = 2\pi - \alpha/3,$$

for which $\Delta t=12/13$ hours and $\alpha=(6/13)\pi$, or

$$(2) \qquad [h_0 + (2\pi/12)\Delta t] - h_0 = 2\alpha/3, \quad (m_0 + 2\pi\Delta t) - m_0 = 2\pi - 2\alpha/3,$$

for which $\Delta t=12/13$ hours and $\alpha=(3/13)\pi$. Hence the shortest time interval is 12/13 hours.

The hands will form the smaller angle $\alpha=(3/13)\pi$ at t hours after 3:00 when $(0+2\pi t)-(\pi/2+2\pi t/12)=(3/13)\pi$, or $t=57/143$. Hence the earliest one could begin the experiment is 57/143 hours, or 23 131/143 minutes, after 3:00 o'clock.

Also solved by Eugene Albert, Leon Bankoff, Merrill Barnebey and F. D. Ceruti (jointly), H. F. Bennett, Julian Braun, E. W. Brown, P. F. Clemens, P. J. Diamandis, Michael Goldberg, S. T. Gormsen, Sidney Kravitz, J. W. Layman, J. L. Leonard, D. C. B. Marsh, Helen M. Marston, L. V. Mead, G. J. Michaelides, and the proposer. Late solutions by C. D. Anderson, J. W. Baldwin, André Dupras, Joe Lipman, J. B. Muskat, and C. F. Pinzka.

Editorial Note. Most of the solvers interpreted the second question of the problem to mean (as in the above solution): How soon after 3:00 can one start an experiment which will lead to trisection *in the shortest possible time*? Braun and Marston interpreted the second question of the problem to mean: How soon after 3:00 can one start an experiment which will ultimately lead to a trisection? Under this interpretation the answer is 1 37/143 minutes after 3:00 o'clock, but one will have to wait 10 2/13 hours for the trisection. A number of the submitted solutions gave entirely different answers.

"I Shall Arise the Same, though Changed"

E 1328 [1958, 526]. *Proposed by Winton Laubach, Colorado School of Mines*

A ray from the origin intersects the circle $\rho=1$ at C and the spiral $\rho=e^{\theta}$, $\theta>0$, at S. Tangents to the circle at C and to the spiral at S intersect at P. Identify the locus of P.

Solution by C. S. Ogilvy, Hamilton College. The arc length of the spiral from the point (1, 0) to S is given by

$$\int_1^{OS} [1 + \rho^2(d\theta/d\rho)^2]^{1/2} d\rho = \sqrt{2}(OS - 1) = \sqrt{2}SC = SP,$$

because SPC is an isosceles right triangle. Thus P lies on that involute of the spiral which intersects it at (1, 0).

Also solved by Eugene Albert, A. P. Boblétt, Stuart Friedman, Michael Goldberg, A. G. Grace, Jr., L. I. Lowell, D. C. B. Marsh, Helen M. Marston, Paul Payette, Ricky Ritterman, D. A. Robinson, E. M. Scheuer, David Zeitlin, and the proposer. Late solutions by C. D. Anderson, J. W. Baldwin, Oldrich Buchta, J. E. Darraugh, S. H. Greene, and Joe Lipman.

Editorial Note. The locus of P is another logarithmic spiral. Its polar equation is $\rho^2=1+(1-e^{\theta})^2$. For an allied problem see E 784 [1948, 317].

Spherical Analogue of the Pythagorean Theorem

E 1329 [1958, 526]. *Proposed by Jose Gallego-Diaz, Vanderbilt University*

A spherical square is a spherical quadrilateral whose four sides are equal and whose four angles are equal. If we let a, b, c denote the areas of the spherical squares constructed on the legs and hypotenuse of a right spherical triangle, show that

$$\tanh^{-1}(\sin c/4) = \tanh^{-1}(\sin b/4) + \tanh^{-1}(\sin a/4).$$

Solution by D. A. Robinson, University of Wisconsin. Let the sphere have radius 1 and let a_1 be the angular measure of the side of the right spherical triangle on which is constructed the spherical square of area a. Each of the four equal angles of this square is $a/4+\pi/2$. By considering the two isosceles spherical triangles which comprise this square and applying the cosine law for angles, one obtains upon solving for $\cos a_1$ and employing standard trigonometric identies

$$\cos a_1 = \frac{1 - \sin a/4}{1 + \sin a/4}.$$

If b_1 and c_1 are defined similarly, analogous expressions are obtained for $\cos b_1$ and $\cos c_1$. By Napier's Rules, $\cos c_1 = \cos a_1 \cos b_1$. Hence

$$\frac{1 - \sin c/4}{1 + \sin c/4} = \frac{1 - \sin a/4}{1 + \sin a/4} \cdot \frac{1 - \sin b/4}{1 + \sin b/4}.$$

Taking the natural logarithm of both sides, multiplying both sides by $-1/2$, and using the fact that

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x},$$

the desired result is obtained.

Also solved by Eugene Albert, D. C. B. Marsh, Paul Payette, Ricky Ritterman, and the proposer. Late solutions by D. R. Brillinger, Joe Lipman, and Benjamin Sapolsky.

An Infinite Product Representing e

E 1330 [1958, 527]. *Proposed by J. L. Pietenpol, Columbia University*

Find the value of the infinite product $\prod_{n=1}^{\infty}(1+1/a_n)$, where $a_1=1$, $a_n=n(a_{n-1}+1)$.

Solution by N. J. Fine, Institute for Advanced Study. The nth partial product is

$$\begin{aligned} P_n &= [(a_1+1)/a_1][(a_2+1)/a_2]\cdots[(a_n+1)/a_n] \\ &= [(a_1+1)/a_2][(a_2+1)/a_3]\cdots[(a_{n-1}+1)/a_n](a_n+1) \\ &= (a_n+1)/n!. \end{aligned}$$

Now

$$\begin{aligned} P_n - P_{n-1} &= [(a_n+1)/n!] - [(a_{n-1}+1)/(n-1)!] \\ &= [(a_n+1)/n!] - [a_n/n!] = 1/n!. \end{aligned}$$

Hence

$$\begin{aligned} P_n &= P_1 + 1/2! + 1/3! + \cdots + 1/n! \\ &= 1 + 1/1! + 1/2! + 1/3! + \cdots + 1/n! \end{aligned}$$

and $\lim_{n\to\infty} P_n = e$.

Also solved by Eugene Albert, R. G. Albert, Merrill Barnebey, David Barr, Julian Braun, D. A. Breault, R. L. Causey, P. L. Chessin, R. J. Cormier, Underwood Dudley, E. L. Ellis, F. A. Ficken, Fred Galvin, Michael Goldberg, A. G. Grace, Jr., Emil Grosswald, J. H. Hodges, J. Hooley, E. H. Kanning III, R. P. Kelisky, D. A. Kearns, P. G. Kirmser, A. G. Konheim, M. I. Knopp and E. H. Scheuer (jointly), Morton Kupperman, Gerald Leibowitz, R. W. McChesney, D. C. B. Marsh, Clifford Marshall, G. I. Michaelides, Leo Moser, Joseph Muskat, C. S. Ogilvy, Paul Payette, Ricky Ritterman, D. A. Robinson, H. D. Ruderman, Jeff Scargle, R. E. Shafer, R. P. Tapscott, W. F. Trench, Chih-yi Wang, Kenneth Williams, David Zeitlin, and the proposer. Late solutions by H. F. Bechtell, D. R. Brillinger, Oldrich Buchta, J. E. Darraugh, D. A. Freedman, S. H. Greene, Walter Penney, C. F. Pinzka, and Benjamin Sapolsky.

Wang showed, more generally, that if $a_n=n(a_{n-1}+z^{n-1})$, where z is any complex number, then the value of the infinite product is e^z.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4834. *Proposed by Oystein Ore, Yale University*

It is well known that the number 30 is the largest integer such that the set of reduced residues (mod 30) includes no compositive numbers. Determine all integers n such that the $\phi(n)$ reduced residues (mod n) are powers of primes.

4835. *Proposed by F. H. Northover, Carleton University, Ottawa, Canada*

Prove

$$\sum_{\nu=0}^{\infty}\binom{n+\nu-1}{\nu}^2 x^\nu = (1-x)^{1-2n}\sum_{k=0}^{n-1}\binom{n-1}{k}^2 x^k.$$

4836. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider the equation $x^2-dy^2=-1$ where d is a nonsquare integer, and suppose that $x=a$ and $y=b$ is any solution in integers. Show that there are infinitely many solutions (x, y) in which x is a multiple of a and y is a multiple of b.

4837. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider the solutions (x, y) of the equation $x^2-dy^2=-1$. Show that every x is relatively prime to every y.

4838. *Proposed by R. L. Duncan, Pennsylvania State University*

Let $A=\{a_n\}$ and $B=\{b_n\}$ denote strictly monotone sequences of positive integers and define the product $AB=\{b_{a_n}\}$. Also, let $\delta(A)=\lim A(n)/n$ denote the natural density of A, where $A(n)$ is the number of elements of A not exceeding n, and let S denote the set of all sequences A for which $\delta(A)$ exists. It has been shown by Niven that if $\delta(A)$ and $\delta(B)$ exist then $\delta(AB)$ exists and $\delta(AB)=\delta(A)\delta(B)$. It is easy to see that S is a semigroup with a unit and that the mapping $A\to\delta(A)$ is a homomorphism of S onto the unit interval. Is this the only such homomorphism? If there are others can they be completely determined and can they be eliminated by requiring that two sequences have the same image whenever they differ in a finite number of places?

4839. *Proposed by R. G. Bushman, University of Wichita*

Let $d(n)$ denote the number of divisors of n, and $[x]$ denote the greatest integer $\leqq x$. Determine the validity of the identity

$$\int_1^x \left(\frac{1}{v}\sum_{n=1}^{[v]} d(n)\right) dv \equiv \int_1^x \left(\frac{[v]}{v}\left[\frac{x}{v}\right]\right) dv, \qquad x \geqq 1.$$

SOLUTIONS

Binomial Theorem in an Associative Algebra

4789 [1958, 370]. *Proposed by Kurt Mahler and P. M. Cohn, the University, Manchester, England*

Let A be an associative algebra over a field of characteristic zero. If $(x+y)^n = \sum_r \binom{n}{r} x^r y^{n-r}$ for some $n \geqq 2$ and for all x and y in A, is it true that $(x+y)^{n+1} = \sum_r \binom{n+1}{r} x^r y^{n+1-r}$ for all x and y in A?

Solution by R. C. Lyndon, University of Michigan. We prove the proposition also for fields of characteristic p (with the assumption $n \leqq p$).

Write $(x+y)^n = x^n + S_1 + \cdots + S_{n-1} + y^n$, a sum of homogeneous parts. The hypothesis gives

$$[nx^{n-1}y - S_1] + \left[\binom{n}{2}x^{n-2}y^2 - S_2\right] + \cdots + [nxy^{n-1} - S_{n-1}] = 0,$$

where each bracketed term is homogeneous. Replacing x in this equation successively by $1x, 2x, \cdots, (n-1)x$ yields a system of equations with nonvanishing Vandermonde determinant. It follows, in particular, that $nx^{n-1}y - S_1 = 0$. Now $S_1 = x^{n-1}y + S'x$, where $S' = \sum x^h y x^k$, $h+k = n-2$. Therefore $(n-1)x^{n-1}y = S'x$, and $(n-1)x^n y = xS'x$. Since $(x+y)^n = (y+x)^n$, the hypothesis is symmetric under interchange of left and right, and therefore by symmetry we have $(n-1)yx^n = xS'x$. It follows that $(n-1)x^n y = (n-1)yx^n$, and that $x^n y = yx^n$, for all x, y. In view of this, $(x+y)^{n+1} = (x+y)(x+y)^n = x(x+y)^n + y(x+y)^n = x(x+y)^n + (x+y)^n y$. Substituting for $(x+y)^n$ the expression given by the hypothesis and collecting coefficients now gives the desired conclusion.

Also solved by L. Carlitz, and the proposers.

A Sum of Legendre Symbols

4790 [1958, 370]. *Proposed by Leonard Carlitz, Duke University*

Let p be a fixed prime >2 and put $\psi(x) = (x/p)$, the Legendre symbol. Evaluate the sum

$$N_r(a) = \sum_{x_1, \cdots, x_r = 1}^{p-1} \psi\{x_1 \cdots x_r(a - x_1 - \cdots - x_r)\}.$$

Solution by Robert Breusch, Amherst College, Massachusetts. We first show that

$$(1) \qquad N_1(a) = \begin{cases} -\psi(-1) & \text{if } a \not\equiv 0, \\ (p-1)\psi(-1) & \text{if } a \equiv 0, \end{cases}$$

where, as in all that follows, congruences are taken modulo p. Obviously $N_1(0) = \sum_{x=1}^{p-1} \psi(-x^2) = (p-1)\psi(-1)$. Now, let $a \not\equiv 0$, $b \not\equiv 0$, $a \not\equiv b$, and let s satisfy $b \equiv s \cdot a$. Then

$$N_1(a) = \sum_{x=1}^{p-1} \psi(x(a-x)) = \sum_{x=1}^{p-1} \psi(sx(sa-sx)) = \sum_{y=1}^{p-1} \psi(y(b-y)),$$

because $y \equiv sx$ again takes all the values from 1 to $p-1$. Thus $N_1(a) = N_1(b)$. But

$$\sum_{a=0}^{p-1} N_1(a) = \sum_{x=1}^{p-1} \psi(x) \sum_{a=0}^{p-1} \psi(a-x) = \sum_{x=1}^{p-1} \psi(x) \cdot 0 = 0.$$

Thus $\sum_{a=1}^{p-1} N_1(a) = -(p-1)\psi(-1)$, and therefore $N_1(a) = -\psi(-1)$ for $a \not\equiv 0$.

We wish to prove that, for $k \geq 1$,

$$(2) \quad N_{2k}(a) = \psi(-1)^k \cdot \psi(a) \cdot p^k; \quad N_{2k-1}(a) = \begin{cases} -\psi(-1)^k \cdot p^{k-1} & \text{if } a \not\equiv 0, \\ (p-1)\psi(-1)^k \cdot p^{k-1} & \text{if } a \equiv 0. \end{cases}$$

The proof follows by induction on k. The definition implies the reduction formula

$$N_r(a) = \sum_{x_r=1}^{p-1} \psi(x_r) \cdot N_{r-1}(a - x_r).$$

The second formula of (2) is true for $k = 1$ by (1). Assume it holds for $k = m$. Then

$$\begin{aligned} N_{2m}(a) &= \sum_{x_{2m}=1}^{p-1} \psi(x_{2m}) \cdot N_{2m-1}(a - x_{2m}) \\ &= \psi(a)\psi(-1)^m p^{m-1}(p-1) + \sum_{x_{2m} \not\equiv a} \psi(x_{2m})\psi(-1)^m(-p^{m-1}) \\ &= \psi(a)\psi(-1)^m p^{m-1}(p-1) + \psi(a)\psi(-1)^m p^{m-1} = \psi^{(a)}\psi(-1)^m p^m. \end{aligned}$$

This argument is valid for $a \equiv 0$ as well as for $a \not\equiv 0$. Furthermore

$$\begin{aligned} N_{2m+1}(a) &= \sum_{x_{2m+1}=1}^{p-1} \psi(x_{2m+1}) \cdot N_{2m}(a - x_{2m+1}) \\ &= \psi(-1)^m \cdot p^m \sum_{x=1}^{p-1} \psi(x(a-x)) \\ &= \psi(-1)^m p^m N_1(a) = \begin{cases} \psi(-1)^{m+1} \cdot p^m \cdot (-1) & \text{if } a \not\equiv 0, \\ \psi(-1)^{m+1} p^m (p-1) & \text{if } a \equiv 0. \end{cases} \end{aligned}$$

The proof is thus complete.

Also solved by N. J. Fine, Emil Grosswald, J. H. Hodges, Joe Lipman, and the proposer.

An Elementary Inequality

4791 [1958, 370]. *Proposed by R. C. Lyness, Blackpool, England*

If α, β, γ are real and $\alpha^3+\beta^3+\gamma^3=0$, prove

$$[\sum (\beta-\gamma)^2][\sum \alpha^4] \geqq [\sum \alpha^2]^3.$$

Solution by Leonard Carlitz, Duke University. Put $s_n=\sum\alpha^n$. Then

$$\Delta = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}^2 = \begin{vmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$$

$$= (3s_2 - s_1^2)s_4 - s_3(3s_3 - s_1s_2) + s_2(s_1s_3 - s_2^2).$$

Since $\sum (\alpha-\beta)^2=2s_2-2\sum \alpha\beta=3s_2-s_1^2$, it follows that

$$\sum (\alpha-\beta)^2 \cdot \sum \alpha^4 - (\sum \alpha^2)^3 = (3s_2 - s_1^2)s_4 - s_2^3 = \Delta + s_3(3s_3 - 2s_1s_3).$$

Therefore, when $s_3=0$, $\sum (\alpha-\beta)^2 \cdot \sum \alpha^4-(\sum \alpha^2)^3=\Delta\geqq 0$.

Also solved by D. M. Brown, A. G. Clark, James Clunie, D. C. B. Marsh, Yoshio Matsuoka, D. S. Passman, and the proposer.

Reducible Linear Differential Operators

4792 [1958, 370]. *Proposed by D. J. Newman and M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Show that the following operators are reducible:

(1) x^nD^{2n}, (2) $x^{2n}D^n$,

and thus solve the differential equations

(1a) $[x^nD^{2n} - \lambda]y = 0$, (2a) $[x^{2n}D^n - \lambda]y = 0$.

Solution by D. C. B. Marsh, Colorado School of Mines. x^nD^{2n} may be factored as $\{xD^2+(1-n)D\}^n$, as is most easily verified after setting $x=e^z$. This yields the set of reduced equations

$$\{xD^2 + (1-n)D\}y = r_iy,$$

where the r_i range over the nth roots of λ. This is one form of the Bessel equation whose solution is familiar. The complete solution of (1a) is therefore

$$y = x^{n/2}\sum_{i=1}^{n}\{A_iJ_n(2r_i^{1/2}\sqrt{-x}) + B_iY_n(2r_i^{1/2}\sqrt{-x})\},$$

where the r_i range as before, A_i and B_i are arbitrary constants. $x^{2n}D^n$ may be factored as $\{x^2D+(1-n)x\}^n$, yielding the set of reduced equations $\{x^2D+(1-n)x\}y=r_iy$, where the r_i range over the nth roots of λ. This equation is

separable, giving the complete solution of (2a) as

$$y = \sum_{i=1}^{n} C_i x^{n-1} e^{-r_i/x}.$$

Also solved by H. E. Stelson and by the proposers.

Editorial Note. The proposers remark that both equations have been solved previously but not so simply. (1a) is due to Lommel, see Watson, *Bessel Functions*, p. 106. (2a) is given in Kamke, *Differential Gleichungen*, p. 540, with references to Steen and Krug.

An Infinite Sequence of Inscribed Polygons

4793 [1958, 451]. *Proposed by M. H. Lietzke and C. W. Nestor, Jr., Oak Ridge National Laboratory*

In *Mathematics and the Imagination*, Kasner and Newman present the following problem. An equilateral triangle is inscribed in a circle of unit radius, another circle is inscribed in the triangle, a square in this circle, *etc.* Continue the procedure, increasing the number of sides of the regular polygon each time by one. As the number of sides of the inscribed polygon increases, the radii of the shrinking circles converge to a definite limiting value. Find this value. The proposed answer (approximately 1/12) seems to be considerably in error. What should it be?

Solution by Julian Braun, Las Vegas, Nevada. Let L be the required limit. Since the ratio of the apothem to the radius of a regular polygon of k sides is $\cos(\pi/k)$ it follows that

$$L = \lim_{n\to\infty} \prod_{k=3}^{n} \cos(\pi/k).$$

Let $\log L = \sum_{k=3}^{n} \log\cos(\pi/k) + R_n$. From graphical considerations it is seen that

$$\left|\int_{n+1/2}^{\infty} \log\cos\frac{\pi}{k}\,dk\right| < |R_n| < \left|\int_{n}^{\infty} \log\cos\frac{\pi}{k}\,dk - \frac{1}{2}\log\cos\frac{\pi}{n}\right|.$$

By setting $n=12$ (*i.e.*, summing ten terms with estimate of the remainder) calculation with the above formulas yields

$$0.114917 < L < 0.114998.$$

Setting $n=22$ yields $0.114938 < L < 0.114951$. One can carry the process further to obtain 0.11494 correct to five significant figures.

The problem is equivalent to one involving circumscribed polygons solved in *School Science and Mathematics*, 1953, pp. 575–6.

Also solved by E. H. Blum, A. P. Boblétt, A. G. Clark, Michael Goldberg, Emil Grosswald, Frank Herlihy, Edgar Karst, A. J. Kokar, T. M. Little, L. V. Mead, K. K. Norton, F. R. Urbanus, Alan Wayne, and the proposers.

Editorial Note. If $\log\cos(\pi/k)$ is replaced by its Taylor series expansion and the order of sum-

mation reversed, we can reduce the result to a form given by several solvers

$$-\log L = \sum_{n=1}^{\infty} \frac{2^{2n}-1}{n} \zeta(2n)[\zeta(2n) - 1 - 2^{-2n}],$$

where ζ is the Riemann zeta function. Using tabulated values (*e.g. Jahnke and Emde*, p. 273) the desired approximation to the limit is obtained.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

An Introduction to Fourier Analysis and Generalized Functions. By M. J. Lighthill. Cambridge University Press, London, 1958. 79 pp. $3.50.

The author's announced purpose is to provide a simple account of the principal and useful facts about Fourier analysis on the line, or what might more properly be called generalized Fourier analysis. Distributions or generalized functions, the term used in the book, play an important if not predominant role in the author's treatment of the subject.

A generalized function is defined in a rather simple way in terms of the sequences which approximate it, in much the same fashion as a real number is defined as an equivalence class of Cauchy sequences of rationals. The resulting objects are then shown to have derivatives of all orders, Fourier transforms, and appropriate formal properties.

One chapter is devoted to the study of those generalized functions which behave like powers of x or $|x|$, and products of these by logarithmic factors. The Fourier transforms of these particular generalized functions are obtained and are tabulated in convenient form. In this part, however, the author is forced to make some rather arbitrary definitions and the theory loses some of its naturalness.

In the next chapter a rather useful and general technique is developed for obtaining asymptotic expressions for the behavior at infinity of the Fourier transforms of well-behaved generalized functions. Trigonometric series are treated in the final chapter as periodic generalized functions on the line.

On the whole the book strikes the reviewer as being somewhat short of its goal of simplicity. Brevity has replaced readability in many instances, and a number of the proofs are too short to be easily grasped; nevertheless the determined reader will gain much from the book.

R. A. KUNZE
Massachusetts Institute of Technology

College Plane Geometry. By Edwin M. Hemmerling. Wiley, New York, 1958. 310 pp. $4.95.

This book is written for college students who have not studied geometry previously and who will probably not elect any further course in mathematics. It covers much more than the elementary concepts and mensuration formulas of plane geometry. Many historical facts are noted to inform the student of the contributions of mathematics to our culture. The process of deductive reasoning and the postulational method are stressed. Particular attention is paid to reasoning in nongeometric situations. Special exercises are included for the purpose of sharpening the students' critical powers.

The material is well selected and well arranged. Photographs which illustrate important geometric concepts are used effectively. An exceptionally large number of well-drawn figures accompanies both text and exercises. The figures are widely spaced on the pages. Many suitable exercises are included. Beginning teachers, or persons using the text for self-instruction, will find the collection of Summary Tests a valuable feature.

The title is too modest. The book contains chapters on inequalities and variation, trigonometry of right triangles and solid geometry. The latter chapter covers all essential definitions and postulates and the statements (without proofs) of basic theorems.

The reviewer noted several minor defects. Readability of the tables of square roots would be improved by the insertion of vertical rulings. Axioms 13 and 14 on inequalities hold only if the multiplier or divisor is positive. The undefined nature of *point* and *line* could well have been emphasized in the text proper rather than being relegated to a footnote.

These minor criticisms are not meant to detract from the overall excellence of the work. The author has written a fine text, and the publisher has prepared it in an attractive form. They are to be commended for preparing a text in plane geometry for college students. This stimulating approach may well serve as a means of retaining plane geometry as a valuable subject of instruction.

H. S. KALTENBORN
Memphis State University

Combinations of Observations. By W. M. Smart. Cambridge University Press, New York, 1958. 253 pp. $6.50.

This book deals with the mathematical formulation of laws of behavior one encounters in various branches of experimental science and is much concerned with the treatment of observations and measures which are subject to accidental errors. One will also find an introduction to the concept of curve fitting.

The content is divided into nine chapters and three appendices.

Chapter I gives a very good and clear discussion of the notion of frequency distributions and the statistical methods for their theoretical and numerical characterization. The normal distribution, its basis and certain integrals con-

nected with it are discussed quite adequately. Sheppards' corrections to moments are presented and it is most gratifying to note that their limitations are indicated.

The next five chapters discuss certain fundamentals of the theory of errors of observation and the principle of least squares. The basic concepts of probability theory and several derivations of the normal law are clearly presented. Measure of precision (reliability) and the relative importance of observations is carefully discussed. It is gratifying to find in this book a discussion of the representation of a frequency distribution by polynomials, by a trigonometric series, by the Gram-Charlier series and by the Pearson system of curves. This, in the reviewer's opinion, is very important. This latter is found in Chapter VII.

Chapter VI concerns itself with equations of condition in several unknowns including equations of different weights, the concept of residuals, and the formal solution of the normal equations. Several interesting examples are given.

Chapter VIII deals with the correction of statistics and the possible rejection of an observation.

Chapter IX gives an adequate introduction to the conception of correlation.

The book is clearly and well written and is highly recommended as a text for a one-semester course, having calculus as a prerequisite, for students in mathematics and the sciences.

FRANK M. WEIDA
The George Washington University

Analytic Geometry. By E. J. Purcell. Appleton-Century-Crofts, New York, 1958. 289 pp. $4.50.

This new analytic geometry text appears (to one who has not yet taught from it) to be a superior, albeit conventional, book. Among its merits are its good format, its clear printing (the theorems being clearly set apart from discussion), and its length. The book is designed to get a student to the calculus as smoothly as possible. It is short enough to give the instructor no trouble in finishing the material in one semester.

There is no attempt to use vectors, nor to give a simultaneous introduction to plane and solid analytic geometry. The chapter on curve sketching is better and more complete than that in many books; for instance, it uses the theorem from higher plane curves on horizontal and vertical asymptotes of an algebraic curve. However, in this chapter a section on cubic curves adds little. Material on curve fitting and on cycloids has been omitted. In the opinion of this reviewer it would have been better to have included these topics and to have omitted the chapter on tangents to curves.

The student should find the text clear, and he should be able to read it with understanding.

ALICE T. SCHAFER
Connecticut College

A Short Course in Differential Equations and *Elementary Differential Equations.* Both by E. D. Rainville. Macmillan, New York, 1958. x+259 pp. $4.50 and xii+449 pp. $5.50.

The first of these two books was designed for a semester course in ordinary differential equations. Its seventeen chapters cover the usual topics exceptionally well with an adequate number of exercises—approximately 1250. The author's aim in trying to exhibit both the techniques for obtaining solutions and the basic ideas and theory behind these techniques is fully realized.

The first seventeen chapters of the second book are identical to the above; nine additional chapters are included in order to make this text suitable for a two-semester course in differential equations. Of special note are the chapters on the Power Series Method and Solutions Near Regular Singular Points. Chapters on Equations of Hypergeometric Type—covering a short introduction to the gamma and factorial functions, the hypergeometric and confluent hypergeometric equations, Bessel's equations, and the polynomials of Legendre, Hermite, and Laguerre—and Orthogonal Sets contain material not included in the first edition. The text concludes with sections on Partial Differential Equations, Fourier Series, and Boundary Value Problems.

Both texts contain excellent treatments of the material considered, especially from the point of view of the student. Explanations are clear and examples are helpful. The difference of only one dollar in price between these two books leaves no doubt in the mind of the reviewer that the Elementary Differential Equations book is preferable under any circumstances.

D. TRIFAN
University of Arizona

Modern Mathematical Methods and Models, Volume 1. Multicomponent Methods. By the Dartmouth College Writing Group. Photolithoprinted, 1958. iii+327 pp. (Any interested person may get a free copy by writing to Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.)

"This is the first of two volumes of experimental-text materials written by the Dartmouth College Writing Group for the Committee on the Undergraduate Program (CUP) of the Mathematical Association of America. These two volumes form the basis of a CUP course for sophomores specially interested in the biological and social sciences. Volume 1 covers the first semester of this course."

Volume 1 consists of four units of approximately equal length. The first unit, on matrix and vector algebras, develops the necessary algebraic language and tools for both volumes. Topics discussed include systems of linear equations and inequalities, inverse and characteristic values of a matrix, and linear and affine transformations.

Unit two is on calculus and finite differences. Extensive use is made in this unit of *A Handbook for Calculus, Difference and Differential Equations* by E. J. Cogan and R. Z. Norman (New York, 1958). Various types of differential and difference equations are solved using the *Handbook*.

The third unit is on multivariable functions, with emphasis on linear and quadratic functions. Topics discussed include linear and quadratic approximations of functions, canonical quadratic forms, and convex sets and functions.

The final unit is on optimization problems. After a discussion of the usual extrema problems of the calculus, extrema problems of a function subject to constraint equations and inequalities are presented. This leads naturally into linear programming, the final topic of this volume.

This book is written in a pleasingly informal way, with many interesting examples from economics, biology, *etc.*, throughout. It is the opinion of the reviewer that the authors have succeeded admirably in performing their assigned task.

RICHARD JOHNSON
Smith College

Introduction to The Theory of Determinants and Matrices. By Edward Tankard Browne. The University of North Carolina Press, Chapel Hill, 1958. 270 pp. $7.50.

The style of this book is that of formally stated theorem followed by proof. The approach is that of the explicit writing out of the components of matrices, and numerical illustrations are frequent. Among the attractive features of the book are the clear exposition (with some exceptions as in the treatment of determinants), the extensive lists of problems and the generally clear printing and organization of material. Many teachers will welcome the quite explicit treatment (illustrated by worked numerical examples) of lambda matrices, elementary divisors and related topics. Some items in the book will be novel and of especial interest for numerical computations with matrices. Thus the signature of a quadratic form is used (p. 136) to determine the number of distinct pairs of conjugate roots of a polynomial and this is applied in the problems with good effect to the study of the cubic and quartic cases.

Although vector spaces are defined (p. 51), there is very little use of the concept and a matrix is rarely regarded as an operator on a space. The concept of dual space appears not to be used at all. For a course serving as a foundation for further study of algebra, this point of view makes the book unsatisfactory despite its positive virtues for the explicit numerical treatment of matrix problems. In a few places the basic concept is not brought out clearly. For example, the Kronecker reduction of a quadratic form is proved with some care, yet the basic idea that the process consists essentially in completing the square is not emphasized.

The material covered includes that common to most current books on matrix algebra and also considers the equivalence of pairs of bilinear forms. The last

two chapters contain an interesting and useful treatment of commutative matrices and systems of differential equations.

The book has enough novelty in content and treatment to deserve a place in every general mathematics library. Teachers of courses in applied matrix algebra may well consider it seriously as a possible text and will certainly want a copy as a source of examples of alternative and often excellent proofs of standard results and of supplementary material usually not included in a first course.

WALLACE GIVENS
Wayne State University

Statistics Manual With Examples Taken From Ordnance Development. By Edwin L. Crow, Frances A. Davis, Margaret W. Maxfield, U. S. Naval Ordnance Test Stations, China Lake, California, 1955. 279 pp. $6.00 (Copies of this manual may be obtained by the public from Technology Division, Office of Technical Services, Department of Commerce, Washington 25, D. C. Order number PB131483.)

This is a manual written internally in the U. S. Naval ordnance test station. It is a "cookbook" outlining techniques of analyzing some "standard" statistical problems. Though the examples are chosen from ordnance development, the manual should be useful to anyone, (including a social scientist, for example), who needs a summarizing handbook of techniques. The manual tends to be old fashioned, ignoring many recent developments of statistical theory which by now are quite accessible for the experimenter. In fact, it seems to be a shorter and more concise version of dozens of elementary statistics books that already exist. Its main emphasis is on "how to do it" rather than on theory. It should prove useful for a statistically sophisticated scientist who needs a compact collection of certain recipes.

Technical Comments. The main topics covered are standard analysis of variance, regression, and goodness-of-fit techniques, and quality control methods. Confidence set procedures are given but the very useful multiple confidence procedures are never mentioned. There is very little about nonparametric procedures. There is some discussion of power (operating characteristic), but there is little suggestion of the applicability of these ideas to the more interesting problems of the manual.

MEYER DWASS
Northwestern University

BRIEF MENTION

The Neutrino. By James S. Allen. Princeton University Press, 1958. vii+168 pp. $4.50.

A survey of the information currently available on the neutrino, including reports on the progress of investigations through March, 1958. Both pre-parity and the newer post-parity experiments are discussed.

Mathematics for Engineers (Part 2, 5th Ed.). By W. N. Rose. Chapman and Hall, London, 1958. xii+403 pp. 25s. (About $3.50).

Library of Mathematics Books. Edited by Walter Ledermann. The Free Press of Glencoe, Ill., 1958. $4.00 per set.

This series of books seems to be intended primarily for cookbook users of mathematics rather than mathematicians.

Differential Calculus. By P. J. Hilton. vii+56 pp. $1.25. *Sequences and Series.* By J. A. Green. viii+78 pp. $1.25. *Linear Equations.* By P. M. Cohn. viii+74 pp. $1.25. *Elementary Differential Equations and Operators.* By G. E. H. Reuter. viii+67 pp. $1.25. Routledge & Kegan Paul, London.

Some Problems in Chemical Kinetics and Reactivity, Volume I. By N. N. Semenov, Translated from the Russian by Michel Boudart. Princeton University Press, 1958. xii+239 pp. $4.50.

One of the Russian translations prepared under contract with the National Science Foundation, reproduced by photo-offset from typewritten copy.

CORRECTION

The book *Notes on Analog-Digital Conversion Techniques*, reviewed in this MONTHLY, vol. 66, 1959, p. 76, should have been listed as a joint publication of Wiley and The Technology Press and the publication date should have been 1958.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SYMPOSIUM ON BOUNDARY PROBLEMS IN DIFFERENTIAL EQUATIONS

A symposium on Boundary Problems in Differential Equations, with especial reference to recent developments in this field, will be held by the Mathematics Research Center, U. S. Army, at the University of Wisconsin, April 20–22, 1959. Invited speakers, about twenty in number, will each present a thirty-minute paper. Both ordinary and partial differential equations will be considered, the emphasis to be upon methods that are potentially adapted to computation.

Among the speakers will be L. Collatz, G. Fichera, L. Fox, W. T. Koiter, J. Schröder, I. N. Sneddon, R. Bellman, G. Birkhoff, H. Bueckner, R. Courant, J. B. Diaz, J. Douglas, K. Friedrichs, P. Garabedian, B. A. Troesch, R. Varga, C. Wilcox, D. Young.

Persons interested in attending the symposium may receive the program and other details by writing to R. E. Langer, Director, Mathematics Research Center, U. S. Army, 1118 West Johnson Street, Madison 6, Wisconsin.

COLLEGE TEACHING AS A CAREER

A pamphlet entitled "College Teaching as a Career," published by the American Council on Education, is available for free distribution to members of the Association and other interested persons. Requests for copies should be sent to: Harry M. Gehman, Secretary-Treasurer, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

PERSONAL ITEMS

University of Cincinnati: Professor A. J. Macintyre, University of Aberdeen, Scotland, is Visiting Research Professor of Mathematics for the academic year 1958–59; Professor I. A. Barnett, on a year's sabbatical leave, is at the Institute for Advanced Mathematics, Dublin, Ireland; Professor G. M. Merriman, head of the department, is on leave of absence the second semester as Visiting Professor of Mathematics, University of North Carolina; Professor Harry Miller is retiring at the end of the current academic year; Mr. Wesley Love is on leave, pursuing an advanced degree.

Mississippi State University: Mr. F. P. Williams, Rear-Admiral (Ret.) U.S.N., has been appointed Instructor; Mr. H. H. Hight has been appointed Temporary Instructor; Assistant Professor Monica Goen has been promoted to Associate Professor; Assistant Professor M. M. Temple has been promoted to Associate Professor; Miss Mary J. Koelz has been promoted to Assistant Professor.

North Texas State College: Associate Professor George Copp has been promoted to Professor; Associate Professor H. C. Parris has been promoted to Professor and Director of the Mathematics Department.

University of Wisconsin-Milwaukee: Dr. Nicolas Artemiadis, University of Paris, has been appointed Assistant Professor; Mr. M. C. Austin, Lecturer at the Imperial College, University of London, England, has been appointed Visiting Lecturer for one year; Mr. W. G. Collar, University of Wisconsin, has been appointed Instructor.

Mr. R. D. Aeder, formerly Applied Science Representative, International Business Machines Corporation, Los Angeles, California, has been appointed Manager, Special Projects Section.

Dr. W. A. Al-Salam, Duke University, has been appointed Assistant Professor at the University of Baghdad, Iraq.

Professor William Bender, University of South Dakota, has accepted the position of Consulting Physicist, Bendix Aviation Corporation, Ann Arbor, Michigan.

Dr. F. G. Brauer, University of Chicago, has been appointed Lecturer at the University of British Columbia.

Dr. R. W. Brown, Oregon State College, has accepted a position as Research Analyst with Boeing Airplane Company, Seattle, Washington.

Assistant Professor R. K. Butz, Colorado State University, has been appointed Associate Professor at Alabama Polytechnic Institute.

Mr. H. S. Christian, Jr., Sentinel Radio Corporation, Evanston, Illinois, has accepted the position of Chief Engineer, Diamond Power Specialty Corporation, Lancaster, Ohio.

Dr. D. R. Clutterham, Convair, Fort Worth, Texas, has accepted the position of Chief Engineer with the Martin Company, Orlando, Florida.

Mr. A. E. Dean, University of North Carolina, has been appointed Assistant Professor at Rollins College.

Dr. F. W. Donaldson, General Electric Company, Philadelphia, Pennsylvania, has accepted the position of Head, Programming and Operations Section, Ramo-Wooldridge, Inc., Fort Huachuca, Arizona.

Mr. J. F. Elliott, Dobyns Bennett High School, Kingsport, Tennessee, has been ap-

pointed Teacher, Main Township High School, Des Plaines, Illinois.

Dr. A. L. Fass, Queens College, has been promoted to Assistant Professor.

Associate Professor W. H. Fuchs, Cornell University, has been promoted to Professor.

Associate Professor Leonard Gillman, on leave from Purdue University 1958–59 on a Guggenheim Fellowship, is a Member at the Institute for Advanced Study.

Mrs. Mildred W. Going, Army Map Service, Washington, D. C., has accepted a position as Associate Mathematician with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Miss Louisa S. Grinstein, Hunter College, has accepted a position as Systems Analysis Engineer with Republic Aviation Corporation, Mineola, New York.

Associate Professor John Gurland, Iowa State College of Agriculture and Mechanic Arts, has been promoted to Professor.

Mr. C. D. Gustafson, Abilene Christian College, has been appointed Instructor of Advanced Mathematics, Nevada Union High School, Grass Valley, California.

Mr. J. V. Hancock, University of Georgia, has been appointed Associate Professor, Wofford College.

Dr. P. W. Healy, Phillips Petroleum Company, Atomic Energy Division, Idaho Falls, Idaho, has accepted a position as Head of the Computing Section, Aerojet-General Nucleonics, San Ramon, California.

Mr. R. T. Heimer, Pennsylvania State University, has accepted a position as Head of the Mathematics Department, Lockhaven State Teachers College.

Dr. Rufus Isaacs, Hughes Aircraft Company, Culver City, California, has accepted a position as Mathematician, Institute of Defense Analysis, The Pentagon, Washington, D. C.

Dr. H. H. Johnson, Stanford University, has been appointed Instructor at Princeton University.

Dr. A. B. Lehman, Case Institute of Technology, has been promoted to Assistant Professor.

Mr. S. R. Lenihan, General Electric Company, Evendale, Ohio, has accepted a position as Mathematician with the University of California Radiation Laboratory, Livermore, California.

Dr. D. B. Lowdenslager, University of Illinois, has been appointed a Lecturer at Princeton University.

Mr. C. W. Marshall has accepted a position as Applied Mathematician with the Institute for Defense Analysis, Washington, D. C.

Associate Professor K. F. McLaughlin, Florida State University, has been appointed Specialist for Appraisal of the Individual, U. S. Office of Education, Washington, D. C.

Mr. Richard McQuillin, Brown University, has accepted a position as Physicist with Bolt, Beranek and Newman, Inc., Cambridge, Massachusetts.

Associate Professor G. W. Medlin, Wake Forest College, has been appointed Head of the Mathematics Department at Stetson University.

Dr. A. B. Novikoff, University of California, Berkeley, has accepted a position as Research Mathematician with the Stanford Research Institute, Menlo Park, California.

Dr. P. A. Penzo, Convair-Astronautics, San Diego, California, is now a Member of the Technical Staff at the Space Technology Laboratories, Los Angeles, California.

Miss Jean E. Sammet, Sperry Gyroscope Company, Great Neck, New York, has been appointed Supervisor, MOBIDIC Programming Section, Programming and Simulation Department of the Data Processing Laboratory, Sylvania Electric Products, Needham, Massachusetts.

Mr. E. D. Schell, Sperry Rand Corporation, has accepted a position as Advisory Mathematician with the International Business Machines Corporation, Yorktown Heights, New York.

Dr. S. H. Schot, University of Maryland, has been appointed Assistant Professor at the American University.

Mr. W. S. Soar, Ballistic Research Laboratory, Aberdeen Proving Ground, has accepted a position as Mathematician with the National Security Agency, Fort George G. Meade, Maryland.

Dr. O. E. Taulbee, Sperry Rand Corporation, Saint Paul, Minnesota, has accepted a position as Mathematician with Lockheed Aircraft Corporation, Marietta, Georgia.

Dr. D. E. Thoro, University of Florida, has been appointed Assistant Professor at San Jose State College.

Mr. F. S. Van Vleck, University of Nebraska, has been appointed Instructor at the Institute of Technology, University of Minnesota.

Assistant Professor D. W. Wall, on leave from the University of North Carolina from February 1, 1959 to January 31, 1960, will be in residence at the University of Michigan on a National Science Foundation Research Grant.

Dr. T. C. Wallace, Iowa State College of Agriculture and Mechanic Arts, has been appointed Staff Member at the Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

Mrs. Elaine G. Lasecki, University of Massachusetts, has been appointed Assistant Professor at Edgewood College of the Sacred Heart.

Assistant Professor E. M. Zaustinsky, San Jose State College, has been appointed Assistant Professor at Santa Barbara College, University of California.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 199 persons have been elected to membership by the Board of Governors on applications duly certified.

ROBERT O. ABERNATHY, M.A. (California, Berkeley) Asst. Professor, Southern University.

OLIVER ABERTH, M.S. (M.I.T.) Grad. Student, University of Pennsylvania.

MURRAY ABRAMSON, M.A. (Syracuse) Mathematician-Programmer, I.B.M., Endicott, New York.

THOMAS W. ACHESON, M.D. (Manitoba) Physician and Surgeon, Vancouver, B.C.

MRS. JEANNE L. AGNEW, Ph.D. (Radcliffe) Asst. Professor, Oklahoma State University.

YOUSEF ALAVI, Ph.D. (Western Michigan) Asst. Professor, Western Michigan University.

C. DEAN ALDERS, M.A. (Colorado S.U.) Instr., Mankato State Teachers College.

JOSEPH ALTINGER, B.S. (Dayton) Teacher, Cathedral Latin School, Cleveland, Ohio.

BERNARD C. ANDERSON, M.A. (Wayne) Teacher, Redford High School, Detroit.

CHARLES A. ANDERSON, B.S. (Pennsylvania S.U.) Grad. Student, Pennsylvania State University.

GLEN D. ANDERSON, A.B. (Drury) Grad. Student, University of Michigan.

STEPHEN A. ANDREA, Student, Oberlin College.

JACK BARR, M.S. (S.U. of Iowa) Instr., Bradley University.

CLARENCE H. BARTHELMAN, M.A. (Harvard) Head of Dept., Worcester Academy.

MRS. FELICE D. BATEMAN, Ph.D. (Michigan) Acting Asst. Professor, University of Illinois.

GERALD A. BECKER, M.S. (S.U. of Iowa) Asst. Professor, San Diego State College.

COL. LAWRENCE D. BELL, B.E., C.E. (N. Carolina S.C.) Asst. Professor, Lamar State College.

CHRISTOPHER BILLINGS, Student, Oberlin College.

RICHARD A. BLADE, Student, University of Colorado.

MRS. KAY W. BLAIR, M.S. (Minnesota) Instr., Macalester College.

JULIUS R. BLUM, Ph.D. (California) Asst. Professor, Indiana University.

MOSES BOSEMAN, JR., M.S. (Atlanta) Instr., Prairie View A. & M. College.

PATRICK J. BOYLE, Student, San Jose State College.

CHARLES A. BROWN, M.A. (Florida) Instr., Florida State University

ALPHONSE BUCCINO, B.S. (Chicago) N.S.F. Fellow, University of Chicago

WILLIAM F. BURDA, Student, Iona College.

JOHN M. BURGER, Ph.D. (Kansas) Head of Dept., Kansas State Teachers College.

LEONARD S. CALLIS, M.A. (Western S.C.) Head of Science Dept., Star Spencer High School, Oklahoma.

LAWRENCE O. CANNON, Teaching Asst., Utah State University.

REV. ROBERT W. CASE, B.A. (St. Anselm's) Grad. Student, Fordham University.

DONALD E. CATLIN, B.S. (Pennsylvania S.U.) Grad. Asst., Pennsylvania State University.

GEORGE E. CHAMPIE, M.A. (Sacramento S.U.) (Instr., Sacramento Jr. College.

HERBERT S. CHARLIP, A.B. (Indiana) Engineer, General Electric Co., Syracuse, New York.

LAWRENCE R. CLIFFORD, Student, Syracuse University.

DANIEL A. CLOCK, M.A. (Illinois) Asst. Professor, Northern Michigan College.

ALICE F. M. CLOUGH, Student, University of British Columbia.

HERBERT E. COHEN, B.S. (C.C.N.Y.) Physicist, Evans Signal Lab., Ft. Monmouth, New Jersey.

FORREST L. COLING, M.S. (Indiana S.U.) Engineer, North American Aviation.

WILLIAM S. COOPER, M.S. (Atlanta) Grad. Student, Atlanta University.

CECIL CRAIG, JR., M.S. (Michigan) Instr., University of Kentucky, Covington.

MARION E. CRIDER, M.S. (Georgia) Chairman of Div., West Georgia College.

ALFRED E. CROFTS, JR., M.S. (Southern Methodist) Instr., Southern Methodist University.

JAMES M. CRON, A.B. (Rutgers) Editor, College Dept., Harper & Brothers Publishers.

DENNY D. CULBERTSON, B.A. (Minnesota) Ensign, U.S.S. Bennington.

HUGO T. P. D'ALARCAO, Student, University of Nebraska.

HENRY P. DECELL, JR., Student, McNeese State College.

KENNETH L. DECKERT, M.S. (Iowa S.U.) Instr., Bethel College.

DOCK B. DEMENT, M.A. (Louisiana S. U.)

ALLEN C. DEMMIN, M.S. (Wisconsin) Chairman of Dept., Middleton High School, Wisconsin.

DONALD J. DESSART, M.S. (Wisconsin) Asso. Professor, State University Teachers College, Oneonta, New York.

JOHN W. DETTMAN, Ph.D. (Carnegie Tech.) Asst. Professor, Case Institute of Technology.

BENEDICT D. DINEEN, M.A. (Fordham) Instr., Iona College.

DELMAR A. DYERSON, B.S. (Texas A. & I. Coll.) Instr., Arizona State College.

ARNOLD R. ESHOO, Student, Manhattan College.

LEE A. EVANS, M.A. (Oklahoma City) Mathematician, Tinker A.F.B., Oklahoma.

LOUIS M. EYERMANN, B.S. (Rose Poly. Inst.) Teacher, duPont Manual Training High School, Louisville, Kentucky.

WILLIAM G. FARIS, Student, University of Washington.

DONALD C. FERGUSON, M.A. (McGill) Teaching Asst., Rutgers University.

JOSEPHINE M. FERGUSON, B.S. (Alberta) Instr., Casper College.

BILL E. FISHER, M.S. (Oklahoma) Principal and Instr., Noble High School, Oklahoma.

HERBERT I. FLEISCHER, B.S. (Brooklyn Coll.) Member of Tech. Staff, Hughes Aircraft Co.

AMBY M. FOLEY, B.A. (Minnesota) Research Mathematician, Crucible Steel Co. of America.

DUNCAN FRASER, M.A.Sc. (British Columbia) Engineer, Home Oil Distributors, Vancouver, B. C.

IRWIN J. FREDMAN, B.A. (Temple) Mathematician, David Sarnoff Research Lab.

ARTHUR H. FREITAG, M.A. (Cornell) Teacher, Jefferson Senior High School, Roanoke, Virginia.

DAVID FRIEDMAN, Student, Cooper Union.

MRS. JUANITA FULLER, B.A. (Carson-Newman Coll.) Teacher, Dobyns-Bennett High School, Kingsport, Tennessee.

F. THOMAS GALLOWAY, M.A. (American) Chief, Bio-acoustic Lab., Walter Reed Army Hospital.

ROBERT T. GARCIA, B.A. (L.A.S.C.) Member of Tech. Staff, Hughes Aircraft Co.

ROBERT B. GARDNER, Student, Princeton University.

HENRI GARNIR, Sc.D. (Liege) Professor, University of Liege.

HELEN E. GINSBERG, M.A. (Wisconsin) Mathematician, David Taylor Model Basin.

MORTON GOLDBERG, Student, University of Michigan.

WILLIAM GOLDSTEIN, M.A. (Pennsylvania) Asso. Professor, State Teachers College, Trenton, N.J.

SAMUEL W. GREENHOUSE, M.A. (George Washington) Chief, Stat. and Math., N.I.M.H., Bethesda, Md.

LOLA B. GREER, M.S. (Denver) Teacher, Moon Jr. High School, Oklahoma.

MRS. NEVA C. GURLEY, M.A. (Oklahoma) Chairman of Dept., Bristow High School, Oklahoma.

JACK H. HAFFERKAMP, M.S. (Bradley) Asst. Professor, Colorado State College.

HOWARD S. HALL, B.S. (Bloomsburg S.T.C.) Instr., Pennsylvania State University.

CHARLES M. HALTOM, M.A. (Texas) Mathematician, Holoman A.F.B.

ROBERT G. HAMMOND, M.S. (Utah) Asst. Professor, Utah State University.

EDWARD V. HARRIS, M.A. (Oklahoma) Instr., East Texas State College.

WILLIAM H. HAUSDOERFFER, Ed.D. (Rutgers) Chairman of Dept., State Teachers College, Trenton, N. J.

JOHN J. HENNINGHAN, Student, Mississippi State University.

BOYD H. HENRY, M.S. (Iowa) Asst. Professor, College of Idaho.

ERVIN H. HIETBRINK, B.A. (Nebraska Wesleyan) Grad. Asst., University of Nebraska.

THOMAS E. HILDICK, JR., Student, Mississippi State College.

RONALD R. HORNBY, B.A. (Nebraska) Grad. Asst., University of Nebraska.

H. HARRISON HUGHES, M.A. (Washington & Jefferson) Asst. Professor, Washington and Jefferson College.

MRS. ALICE E. A. P. HUNT, Ph.D. (New York) Asst. Professor, Wilson College.

HURSHELL H. HUNT, B.S. (P.A.M.C.) Grad. Asst., Oklahoma State University.

REGINALD C. JACKS, M.A. (Chicago) Asso. Professor, University of Alberta.

ALVIN J. JANIS, B.S. (DePaul) Chairman of Dept., De La Salle Institute.

PAUL S. JORGENSEN, M.S. (Wisconsin) Instr., Carleton College.

ROBERT KALECHOFSKY, B.S. (C.C.-N.Y.) Asst. Professor, State University College on Long Island.

ALBERT S. KAMP, Student, St. Ambrose College.

ARTHUR J. KARSON, Student, University of Tulsa.

WILLIAM Jo KENNEDY, JR., Student, Oklahoma State University.

DONALD G. KILLIAN, M.A. (Missouri) Instr., University of Wichita.

JOSEPH D. KISSINGER, B.S. (Pennsylvania S.T.C.) Grad. Student, Bucknell University.

COL. PIERRE A. KLEFF, M.S. (Holy Cross) Deputy Commander, Rocky Mountain Arsenal.

GEORGE C. KLEIN, B.S. (St. Michael's) Teacher, Cathedral High School, Lafayette, Louisiana.

ARNOLD M. KUZMACK, Student, Harvard University.

COSTAS J. LABOVITES, A.B. (Clark) Mathematician, Army Map Service.

PETER A. LACHENBRUCH, B.A. (U.C.L.A.) Grad. Asst., Lehigh University.

WAYNE LANIER, JR., Student, Oklahoma State University.

GORDON E. LATTA, Ph.D. (California Inst. Tech.) Asso. Professor, Stanford University.

ANNIE L. LAURER, Student, Oberlin College.

BERT LEVY, B.S. (Wilson T.C.) Mathematician, Army Map Service.

GILBERT LIEBERMAN, M.A. (Columbia) Mathematician, U. S. Naval Ordnance Lab.

TUNG-PO LIN, Ph.D. (M.I.T.) Research Chemist, DuPont Experimental Station.

PAUL E. LIVERMORE, M.A. (Arizona S.C.) Asst. Professor, Arizona State College.

CHARLES W. LYTLE, M.S. (New York) Acting Chairman of Dept., Drew University.

MARION L. MACQUEEN, Ph.D. (Chicago) Professor, Southwestern at Memphis.

WILLIAM R. MAINS, A.B. (San Diego S.C.) Teacher, San Diego State College.

RICHARD T. MALAFA, Student, Independence High School, Ohio.

RALPH J. MARSHALL, Student, University of Buffalo.

LYSLE C. MASON, M.S. (Michigan) Professor, Phillips University.

M. ASTON MASON, Student, University of British Columbia.

ROBERT D. MASON, JR., Student, American University.

E. HOWARD MATTHEWS, Ph.D. (Geo. Peabody) Asso. Professor, Southwest Missouri State College.

STEVEN W. MATTHYSSE, Student, Yale University.

JOHN B. MCILROY, JR., M.A. (Columbia) Asst. Professor, State Teachers College, Trenton, N. J.

LEONARD J. MCPEEK, Student, University of British Columbia.

RALPH D. MCWILLIAMS, Ph.D. (Tennessee) Instr., Princeton University.

JACK R. MEAGHER, M.A. (Michigan) Asso. Professor, Western Michigan University.

MRS. ANN S. MILLER, B.A. (Syracuse) Teacher, Roanoke Catholic High School, Virginia.
IRWIN MILLER, Ph.D. (Virginia Poly. Inst.) Asso. Professor, Arizona State College.
DOUGLAS H. MOORE, M.A. (U.C.L.A.) Asst. Professor, California State Polytechnic Institute.
WILLIAM L. MORRIS, B.A. (Cincinnati) Teaching Asst., University of Cincinnati.
DAVID C. MUIR, Student, University of British Columbia.
HAROLD MUKAMAL, Student, Cooper Union.
JOHN EDMONDS NEILL, A.B. (Washington & Lee) Asso. Manager, College Dept., Henry Holt & Co.
DAVID NELSON, Ph.D. (Wisconsin) Asso. Professor, George Washington University.
HENRY M. NODELMAN, M.A. (Yale) Head of Dept., R.C.A. Institute.
ERIC A. NORDGREN, B.Ch.E. (Poly. Inst. of Brooklyn) Engineer, General Electric Co.
THOMAS D. OWEN-TURNER, General Commercial Engineer, British Columbia Telephone Co.
MRS. JEAN J. PEDERSEN, M.S. (Utah) Grad. Student, University of Utah.
SANTO D. PRATICO, Student, Iona College.
RICHARD L. PRICE, M.A. (Texas) Instr., Prairie View A. & M. College.
RONALD L. PRICE, Student, Oberlin College.
ERVIN J. PROUSE, Ph.D. (California, Berkeley) Asso. Professor, University of Texas.
DAVID L. PUGH, Student, Oberlin College.
THOMAS G. PULLEN, III, B.S. in E.E. (Pennsylvania S.U.) Engineer, Western Electric Co.
COLBERT T. PURVIS, Ph.D. (Geo. Peabody) Asst. Professor, Georgia Institute of Technology.
MICHAEL RABBAT, B.S. (Inst. of Tech.) Teacher, De La Salle Secondary School, Cairo, Egypt.
WARREN G. RANDLE, Student, Oregon State College.
RICHARD M. RAU, M.S. (Illinois) Asst., University of Illinois.
DAVID S. RAY, M.A. (Michigan) Teaching Asst., University of Tennessee.
WILLIAM D. RAY, M.S. (Alabama Poly. Inst.) Asst. Professor, Arizona State College.
WAYNE R. RICH, M.S. (Utah S.U.) Asst. Professor, Utah State University.
MRS. VIRGINIA M. T. RICHARDSON, M.A. (Bucknell) Lecturer, Central State College.
BASIL D. ROSENBERRY, M.Ed. (St. Bonaventure) Instr., Amherst Central Senior High School, New York.
MRS. DAISY LEE ROWE, B.A. (Southeastern S.T.C.) Teacher, Duncan High School, Oklahoma.
W. NEAL RYBICKI, Student, San Diego State College.
JOHN J. SACCOMAN, Student, Seton Hall University.
MRS. NAOMI R. SALAMAN, Student, Colorado College.
AIMO SALENIUS, B.S. (Hofstra Coll.) Instr., Westchester Community College.
SALVATORE W. SALERNO, M.A. (Columbia T.C.) Teacher, Barringer High School, Newark, New Jersey.
V. L. N. SARMA, M.S. (Banaras Hindu, India) Grad. Student, University of Rochester.
HAROLD J. SCHEINHAUS, Student, City College of New York.
MRS. MARY B. SCHOENWALD, M.A. (Northwestern) Teacher, Ponca City Senior High School, Oklahoma.
GLYNN D. SEARL, Student, University of British Columbia.
MRS. ROSE W. SEDGEWICK, Ph.D. (Brown) Instr., University of Maryland.
EDWARD J. SELIGMAN, M.A. (Wayne) Computer Systems Analyst, R.C.A.
DANIEL K. SHIMIZU, Student, University of British Columbia.
ROBERT M. SIEGMANN, B.S. (Coll. of Charleston) Grad. Student, University of South Carolina.
RICHARD W. SIELAFF, M.S. (Illinois Inst. Tech.) Divisional Asst., The Northern Trust Co.
SISTER ELIZABETH LOUISE, M.A. (Boston) Instr., Emmanuel College.
SISTER HELEN CLARE FITZSIMMONS, M.S. (Notre Dame) Asso. Professor, Webster College.
SISTER MARY DE LOURDES, B.A. (Creighton) Teacher, Servite High School, Detroit, Michigan.
JAMES D. SMALLWOOD, M.A. (N. Texas S.C.) Mathematician, Chance Vought Aircraft.
D. BRENDAN SMITH, M.A. (Oxford) Head Teacher, St. Louis Priory School, Creve Coeur, Missouri.
WALTER A. SPIVEY, Ph.D. (North Carolina) Asst. Professor, University of Michigan.
ROY O. SPRUIELL, Student, New Mexico College of A. & M.A.
CAPT. MICHAEL J. STEGER, B.S. (U.S.M.A.) U.S.A.F.; Grad. Student, Texas A. & M. College.
ROBERT W. STRINGALL, Student, San Jose State College.
HILDA M. SULLIVAN, Ed.M. (Rochester) Head of Dept., Pittsford Central School, New York.
J. RICHARD SWENSON, Student, Massachusetts Institute of Technology.
RUSSELL J. TECKLENBURG, Student, Walnut Hills High School, Ohio.
WILLIAM A. THOMPSON, JR., Ph.D. (North Carolina) Asso. Professor, University of Delaware.
HAROLD THORNTON, B.S. (London) Senior Math. Master, King Edward VII School, Norfolk, England.
DON H. TUCKER, Ph.D. (Texas) Asst. Professor, University of Utah.
JOSEPH A. VELLOZZI, M.A. (Columbia) Chairman of Dept., Yonkers High School, New York.
L. V. VENKATARAMAN, M.S. (Banaras Hindu, India) Lect., Banaras Hindu University.
E. HERBERT WAENTIG, B.A. (Princeton) Editor, College Dept., McGraw-Hill Book Co., Inc.
THOMAS C. WALES, JR., A.B. (Harvard) Instr., St. Mark's School, Southboro, Massachusetts.
DONALD WARNCKE, M.A. (Columbia) Instr., Monmouth College.
CHARLES F. WEDEN, B.A. (Harvard) Asso. Head, College Dept., D.C. Heath & Co.
WALTER R. WESTPHAL, M.A. (Columbia) Asst. Professor, State University of New York College for Teachers, Buffalo.
JOHN C. WHITE, M.A. (Texas) Asst. Professor, Pacific University.
BRO. FIRMIN G. WIDMER, M.A. (Ohio S.U.) Teacher, Chaminade High School, Dayton, Ohio.
JAMES WIGMORE, M.S. (Purdue) Asso. Scientist, Avco Mfg. Corp.
RICHARD H. WINKLER, Student, Massachusetts Institute of Technology.
FRED A. WOMACK, JR., Student, Texas Christian University.
CRAIG C. WORK, B.A. (Wisconsin) Grad. Asst., Stanford Univeristy.
LLOYD E. YATES, Student, University of Colorado.
JAMES F. ZANE, Student, University of Colorado.
IRVING ZUCKER, Student, McGill University.

THE OCTOBER MEETING OF THE IOWA SECTION

The Iowa Section of the Mathematical Association of America held its second annual fall meeting on Friday, October 17, 1958 at the State University of Iowa, Iowa City. This was held in conjunction with the 28th Annual Conference for Teachers of Mathematics in Iowa. Approximately 185 teachers attended the conference, which included 23 members of the Association.

The theme of the conference was: *Providing for the superior student.* Professors M. F. Smiley and H. V. Price, State University of Iowa, presided at the morning and afternoon sessions respectively.

The program consisted of three invited lectures in the morning and a panel discussion in the afternoon. The morning addresses were: *What the High Schools are doing for the*

superior pupils—the Evanston picture, by Mr. F. P. May, Evanston Township High School, Evanston, Illinois; *A model seminar for superior junior high school pupils in science and mathematics*, by Professor R. A. Nielsen, Iowa State Teachers College, Cedar Falls; and *A model seminar for superior senior high school pupils in mathematics*, by Professor W. W. Gutzman, University of South Dakota.

The subject, *The collegiate honors program in mathematics*, was discussed by a panel consisting of Professor Rhodes Dunlap, State University of Iowa, Moderator, Professor L. C. Graue, Coe College, Professor W. W. Gutzman, and Professor H. T. Muhly, State University of Iowa.

E. N. OBERG, *Chairman*

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held on October 11, 1958 at the University of Minnesota, Duluth Branch, Duluth, Minnesota. Professor J. E. Hafstrom of the University of Minnesota, Duluth, presided at the morning session. The section chairman, The Rev. W. C. Kalinowski, O.S.B., of St. John's University, presided at the afternoon session. There were 53 persons in attendance, including 32 members of the Association.

At the business meeting Professor Robert Sloan of Carleton College reported on the activities of the Committee on High School-College Relations. As directed by a vote of the section at the previous meeting, this committee has informed the Minnesota State Department of Education of the availability and willingness of members of the Association to serve as invited consultants at high schools which are in the process of re-evaluating their mathematics programs. Several other projects of this committee were discussed but are in a formative stage. Professor G. K. Kalisch of the University of Minnesota reported on the 1958 high school mathematics contest sponsored by the section and described plans for the 1959 contest. It is planned to finance the 1959 contest through the newly formed Industry-Education Board of the Minnesota Academy of Science.

Professor K. W. Wegner of Carleton College, who was the section representative at the summer meeting of the Association, reported on that meeting.

The following papers were presented:

1. *Asymptotic solution of an implicit function*, by Professor G. A. Heuer, Concordia College.

Let an experiment with N equally likely outcomes be performed K times. The probability of a repeated outcome is $P(N, K)=1-N!/(N-K)!N^K$. Denoting this number by t, the problem of solving for K as a function of N and t is considered. THEOREM. *For* $0<t<1$, K *is asymptotic to* $\sqrt{-2\log(1-t)}\sqrt{N}$. Machine computation shows the approximation to be good for small N.

2. *On numerical evaluation of series*, by Professor O. E. Stanaitis, St. Olaf College.

The sum of the first k terms, say $S_k=\alpha$ is calculated. The remainder is evaluated by Euler's summation formula, say $\gamma<R_k<\beta$. Hence, the approximation $\gamma+\alpha<S<\beta+\alpha$ is obtained. It has been shown that for the series $S=\sum 1/n^2$ it would be necessary to calculate a hundred terms in order to secure accuracy to 2 decimal places. By the above technique the same accuracy is reached by adding only six terms of the series.

3. *Mathematics education in Italy*, by Dr. Amos Nannini, Visiting Lecturer at the University of Minnesota, Duluth. (By invitation of the Executive Committee.)

Education in Italy is very much the same all over the country as there is only one government, in Rome. Italian educational institutions follow the 5-3-5 system. Education is free and compulsory up to the age of fourteen. Youngsters at the age of 14 start receiving instruction in rational geometry and in algebra.

In about 1900, a group of eminent mathematicians, led by Professor Federigo Enriques, intro-

duced more rigor into geometry as it is taught, and discussed some topics of mathematical interest in the elementary field.

Italian college students have a fairly good background in mathematics, and college standards are high.

4. *A program for advanced standing in mathematics for high school students*, by Professor W. B. Stenberg, University of Minnesota, introduced by Professor H. L. Tumittin.

The program at the University of Minnesota was described.

5. *The Minnesota program of educational experimentation in mathematics*, by Professor P. C. Rosenbloom, University of Minnesota.

The School Mathematics Study Group and the Minnesota National Laboratory for the Improvement of Secondary School Mathematics were described. The materials produced at the writing conference at Yale last summer were outlined. The designs of the 7–8th grade and the 9th grade experiments to be performed this year, were also described. The cooperation of college mathematicians was requested to provide appropriate courses for prospective teachers and teachers in service in order to increase the number of teachers qualified to participate.

F. L. WOLF, *Secretary*

OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University on October 24, 1958. The fall meeting of the Oklahoma Section is held in conjunction with the Oklahoma Education Association, and is devoted to papers of particular interest to high school teachers. Research papers are presented in the spring meeting.

In the absence of the Chairman and the Vice-chairman, Professor R. V. Andree, the Secretary, presided. There were 299 persons in attendance, including 72 members of the Association. The following officers were elected for one-year terms: Chairman, Professor D. P. Richardson, University of Arkansas; Vice-chairman, Professor Kathrine Mires, Northwestern State College; Secretary-treasurer, Professor R. V. Andree, University of Oklahoma.

The Association's Representative to the Junior Academy of Science, Professor R. B. Deal, reported that 50 papers had been submitted by high school students, and from this group 25 papers were selected for the program.

The Traveling Lecturer Committee, under the chairmanship of Professor Kathrine Mires, reported that progress was being made in establishing a lecture bureau for high schools in the state of Oklahoma. Professor D. P. Richardson, liaison officer for the Oklahoma Section of the Association with the State Academies of Science, reported that the Academies of Science of the states of Arkansas and Oklahoma and the Oklahoma Section of the Association would soon submit a proposal to the National Science Foundation for the sponsorship of a visiting lecturer program in the two states. Professor Carter, chairman of the High School Contest Committee, reported that plans were underway for Oklahoma and Arkansas to participate even more actively during the coming year in the Mathematical Association of America's Contest than they did last year.

The Secretary reported upon the national meeting of sectional officers.

The following program was presented as being of particular interest to high school teachers. The enthusiasm with which it was received seems to indicate that the topics were timely.

The concept of number, Professor Gene Levy, University of Oklahoma.
Functions, Professor R. B. Deal, Oklahoma State University.
Operating with sets, Mr. A. S. Davis, University of Oklahoma.
Point-Set topology, Professor O. H. Hamilton, Oklahoma State University.

The scientist—man of destiny, (luncheon program), Mr. H. M. Smith, Regional Director, U. S. Bureau of Mines, introduced by the Secretary.

The hypercircle in modern physics, Professor C. E. Springer, University of Oklahoma.

Statistics and data handling for high schools, Professor J. C. Brixey, University of Oklahoma.

Limits in geometry and algebra, Professor Arthur Bernhart, University of Oklahoma.

R. V. ANDREE, *Secretary*

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The annual meeting of the Northeastern Section of the Mathematical Association of America was held at the College of the Holy Cross, Worcester, Massachusetts on November 29, 1958. Professor D. E. Richmond, Chairman of the Section, presided. There were 115 persons registered for the meeting, including 90 members of the Association.

Officers chosen for 1958–59 were: N. H. McCoy, Smith College, Chairman; J. G. Kemeny, Dartmouth College, Vice-Chairman; V. O. McBrien, College of the Holy Cross, Secretary-Treasurer. The Section voted to sponsor the National High School Mathematics Contest and Rev. S. J. Bezuszka, S. J., Boston College, was appointed chairman of the Section Contest Committee.

The following members of the Association were appointed to act as liaison officers to deal with mathematical matters within their respective states or provinces: C. T. Holmes, Bowdoin College, Maine; J. B. Adkins, Phillips Exeter Academy, New Hampshire; D. H. Ballou, Middlebury College, Vermont; A. A. Bennett, Brown University, Rhode Island; J. W. Bower, Connecticut College, Connecticut; W. J. Blundon, Memorial University, Newfoundland; Rev. E. J. Roche, St. Dunstan's University, Prince Edward Island; W. S. H. Crawford, Mt. Allison University, New Brunswick; J. J. Adshead, Dalhousie University, Nova Scotia.

By invitation of the Section, the following papers were presented:

1. *The report of the Commission on Mathematics*, by Professor A. W. Tucker, Princeton University.

2. *The School Mathematics Study Group*, by Professor D. E. Richmond, Williams College. (Professor Richmond spoke in place of Professor E. G. Begle of Yale University who was unable to attend the meeting.)

3. *The concept of a "truth set,"* by Professor J. G. Kemeny, Dartmouth College.

The concept of a "truth set" may be used as a unifying concept in the presentation of many branches of mathematics. Given a set of logical possibilities, we may associate with any statement concerning these possibilities a subset of the given set. The truth set of the statement p is the set P of all possibilities in which the statement is true. This establishes a natural isomorphism between logic and set theory, and many applications may be given. Applications to linear equations and inequalities, to systems of functional equations, and to probability theory are discussed.

4. *Incidence geometry—a unifying concept*, by Professor Walter Prenowitz, Brooklyn College.

The notion of incidence geometry is suggested by Hilbert's set of Postulates of Incidence in his *Foundations of Geometry*. The concept and name are due to Gorn, Bull. Amer. Math. Soc. vol. 46, 1940, pp. 158–167. In the present paper an incidence geometry is characterized as a system (S_1, S_2, S_3) composed of three sets whose elements are called respectively *points*, *lines*, *planes* which satisfy a set of postulates essentially equivalent to that of Hilbert. A few elementary theorems are stated, and a list of ten incidence geometries is given. How to characterize most familiar types of geometry as incidence geometries is indicated.

5. *Recursive functions and set theory*, by Professor Hartley Rogers, Massachusetts Institute of Technology.

An expository discussion is given, presupposing a minimum of special background, of recursive function theoretic constructions on the integers that provide countable analogues to various transfinite structures of classical set theory. References include work of Dekker and Myhill on constructive versions of cardinal number theory, and work of Church, Kleene, Markwald, Spector, Kreider and the author on constructive versions of ordinal number theory.

V. O. McBRIEN, *Secretary*

CORRECTION

In the report of the March 1958 meeting of the Michigan Section and in the index (this MONTHLY, vol. 65, 1958, p. 551 and p. 810, respectively) the name P. J. Nikolai was misspelled "Nakolai."

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Valparaiso University, May 2, 1959.

IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.

KANSAS, Marymount College, Salina, April 11, 1959.

KENTUCKY, Centre College of Kentucky, Danville, April 25, 1959.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, May 2, 1959.

METROPOLITAN NEW YORK, Polytechnic Institute of Brooklyn, April 18, 1959.

MICHIGAN, Michigan State University, East Lansing, March 28, 1959.

MINNESOTA, University of Minnesota, Minneapolis, April 25, 1959.

MISSOURI, Lindenwood College, St. Charles, April 25, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Miami University, Oxford, May 9, 1959.

OKLAHOMA, Tulsa University, April 10–11, 1959.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN, Utah State University, Logan, May 8–9, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State University, Tempe, April 10–11, 1959.

TEXAS, University of Texas, Austin, April 17–18, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May 2, 1959.

Watch for

→ **FUNDAMENTAL PRINCIPLES OF MATHEMATICS**

by John T. Moore

UNIVERSITY OF FLORIDA

Admirably meets the latest recommendations for the teaching of college mathematics, with outstanding clarity and simplicity

Contents: The Basic of Sets. The Real Number System. An Aid to Computation. Relations and Functions. The Basic Elementary Functions. Some Basic Principles of Graphing. Further Development and Applications of Circular Functions. A Glimpse of Differential Calculus. Sequences and a Glimpse of Integral Calculus. Polynomial (and other) Equations and Inequalities. Systems of Equations. Plane Analytic Geometry: 1. Curve Sketching; 2. Polar Coordinates; 3. Locus Problems. Solid Analytic Geometry. Permuations, Combinations, and the Binomial Theory. Probability and Statistics.
Appendix: Review of High School Algebra. Tables.

Ready for your inspection in April. Send for your copy now.

from the publishers of

→ **Introduction to the Foundations & Fundamental Concepts of Mathematics**

by Howard Eves & Carroll V. Newsom
363 pages, $6.75

Texts specifically noted and highly recommended at the Christmas Meeting of the N.C.T.M.

Understanding Arithmetic

by Robert L. Swain
264 pages, $4.75

→ **Rinehart Mathematical Tables, Formulas & Curves**

by Harold Larsen
280 pages, $2.50

The tables preferred by mathematicians everywhere for their accuracy, completeness and clarity

Rinehart & COMPANY, INC. • 232 MADISON AVE., NEW YORK 16, N.Y.

AN OUTSTANDING MATHEMATICS TEXT

A Course in Mathematical Analysis, Volume 1

By NORMAN B. HAASER, JOSEPH P. LASALLE, and JOSEPH A. SULLIVAN, University of Notre Dame

This book presents introductory analysis from the point of view and in the spirit of contemporary mathematics. The authors have, however, given ample attention to the language and symbolism of classical analysis so as to familiarize the student with these. Volume II, in preparation, will contain an extension of the geometry and calculus in Volume I.

GINN AND COMPANY Home Office: Boston

Sales Offices: New York 11, Chicago 6, Atlanta 3, Dallas 1, Palo Alto, Toronto 16

presentation is clear and compact
exercises are varied and abundant
in

BASIC MATHEMATICS:
A Workbook Forms A and B
M. WILES KELLER
JAMES H. ZANT

Form A 253 pages Paper covers $2.20
Form B 255 pages Paper covers $2.20

BASIC ALGEBRA
M. WILES KELLER
290 pages Paper covers $2.50

COLLEGE ALGEBRA
2nd edition
M. WILES KELLER
471 pages $4.50

HOUGHTON MIFFLIN COMPANY

Boston 7 Massachusetts
New York 16 New York
Atlanta 5 Georgia
Geneva Illinois
Dallas 1 Texas
Palo Alto California

PLANE TRIGONOMETRY

THIRD EDITION

by

RAYMOND W. BRINK

In this revision, the modern tendency toward increased emphasis on analytical trigonometry as distinguished from the computational, is recognized. The analytical topics have been moved to the early chapters of the book and given more extensive treatment than previously. Some new, more analytical, topics have been introduced, such as sets and functions and functional notation, and new more analytical proofs have been provided. Throughout the book there is an imediate application of principles to problems, either as illustrative examples solved in the text or as exercises for the student. The text is especially adaptable for college courses of various lengths and purposes.

APPLETON CENTURY MATHEMATICAL SERIES

Just Published

APPLETON-CENTURY-CROFTS, INC.

35 West 32nd Street, New York 1, New York

OPERATIONS RESEARCH ANALYSTS:

Computer Department has several openings for well-qualified operations research analysts to participate in planning and developing complex optimization models, dynamic simulation programs and integrated accounting applications for large tape computer. Candidates must meet following academic and experience requirements:

ANALYST OR ECONOMIST:

Degree in engineering, physical science or economics preferably with math minor, plus experience with refinery process analysis, transportation economics or marketing and distribution.

MATHEMATICIAN OR STATISTICIAN:

Ph.D. in mathematics or statistics with strong background in numerical analysis, matrix algebra or econometrics. Experience with computers desirable, but not essential.

TIDEWATER OIL COMPANY

World Headquarters

4201 Wilshire Blvd., Los Angeles 5, Calif.

Coming in May from
ADDISON-WESLEY

BY GEORGE B. THOMAS, JR.

ELEMENTS OF CALCULUS AND ANALYTIC GEOMETRY

Designed for a one-year course, this new text is based on the author's widely used *Calculus and Analytic Geometry*, and contains most of the material from that book on the calculus of functions of one independent variable, and on plane analytic geometry. Much of the material has been revised and reorganized, but the level of rigor has been maintained. Since it retains the modern treatment, the early introduction to integration, and the careful, thorough manner of presentation which characterized Professor Thomas' other book, it should be well suited for shorter or less concentrated courses offered in the colleges.

In addition, *Elements of Calculus and Analytic Geometry* should be particularly useful for accelerated courses offered in the secondary schools, such as the Advanced Placement Program, or for college-preparatory programs offered to students who are going on to study mathematics at the college level. The need for such a text is evidenced by the fact that currently more than 40 public and private high schools are using *Calculus and Analytic Geometry*, even though the amount of material it contains is beyond the syllabus which most of these schools wish to cover in a one-year course. *Elements of Calculus and Analytic Geometry* thus offers a textbook of college level, but with the selection of material suitable for high-school programs.

c. 576 pp, 315 illus, 1959—$7.50

also by Professor Thomas

Currently the adopted texts in 221 colleges and universities all over the world

CALCULUS *692 pp. 333 illus, 1953—$7.50*

CALCULUS AND ANALYTIC GEOMETRY

822 pp, 418 illus, 2nd ed. 1953—$8.50

ADDISON-WESLEY PUBLISHING COMPANY, INC.
Reading, Massachusetts, U. S. A.

Texts for advanced courses from
ADDISON-WESLEY

AN INTRODUCTION TO PLASTICITY

By WILLIAM PRAGER, *Brown University*

This book is based on a series of lectures that the author gave at the Federal Polytechnic Institute in Zurich, which were originally published in the German language in a Swiss edition in 1955. It is designed to supplement traditional texts on strength of materials and theory of structures, which are primarily concerned with elastic behavior. By restricting the scope of the book to selected topics from the theory of perfectly plastic solids, the author is able to bring the reader up to the frontiers of current research without presupposing prior familiarity with the subject.

c. 128 pp, 82 illus, to be published Fall 1959—$6.50

INTRODUCTION TO RIEMANN SURFACES

By GEORGE SPRINGER, *University of Kansas*

A modern, self-contained treatment of the fundamental concepts and basic theorems concerning Riemann surfaces, which assumes a knowledge of elementary complex function theory and some real variables and algebra. The book gives the student excellent preparation for further study of current progress in Riemann surfaces, the modern theory of manifolds, algebraic geometry, and topology. Although it is intended for a two-semester course, it may also serve as a text for a shorter course, by proper selection of material.

307 pp, 78 illus, 1957—$9.50

INTRODUCTION TO MEASURE AND INTEGRATION

By M. E. MUNROE, *University of Illinois*

The aim of this textbook is to present measure theory from the abstract or postulational point of view and yet to do this in such a way that the student as well as the expert will find the work helpful. The main outline of the theory is developed in unstarred sections of the book. Many interesting applications of measure theory to other topics in mathematics are given in starred sections.

310 pp, 14 illus, 1953—$8.50

ANALYTIC AND PROJECTIVE GEOMETRY

By D. J. STRUIK, *Massachusetts Institute of Technology*

291 pp, 116 illus, 1953—$7.50

DIFFERENTIAL GEOMETRY

By D. J. STRUIK

221 pp, 121 illus, 1950—$7.50

Two clearly written and graphically illustrated texts by an outstanding geometer.

ADDISON-WESLEY PUBLISHING COMPANY, INC.
Reading, Massachusetts, U. S. A.

important Harper texts

H B

FOUNDATIONS of MATHEMATICS

CARL H. DENBOW & VICTOR GOEDICKE, *Ohio University*

new

Based on careful study of the needs of modern mathematical instruction, this is an unusually adaptable new text providing a broad coverage of the philosophy of mathematics (and of advanced topics of special interest) in addition to utilitarian mathematics through calculus. Research leading to the development of this book was carried on while both authors were Fellows of the Ford Foundation's Fund for the Advancement of Education at Harvard (1955-56). Preliminary drafts of this text were used for several years in National Science Foundation Institutes for Teachers of Mathematics. 620 pp. $6.00.

VECTOR ANALYSIS

With Applications to Geometry and Physics

MANUEL SCHWARTZ, *University of Louisville*, and SIMON GREEN & W. A. RUTLEDGE, *University of South Carolina*

forthcoming

An outstanding book by three distinguished professors. Designed to facilitate the understanding of both the fundamental concepts and the techniques of vector analysis, with sufficient applications to render it a useful tool in physics, engineering, and mathematics. Numerous examples and problems, from simple to complex, coupled with the wide range of applications encourage use on a self-study basis. *May.*

COLLEGE ALGEBRA

Second Edition

THURMAN S. PETERSON, *Portland State College*

recent

A modernized and revised (the book has been entirely rewritten and reset) edition of a famous Peterson text designed to serve as a foundation course for those preparing for more advanced college mathematics, *and* as a terminal course for students preparing for nonscientific studies. Revision features many new topics currently in demand in college algebra courses, expansion of exercises with new and modern problem situations, a clear-cut division into the "intermediate algebra" course (Chapters 1-12) and the "college" course (Chapters 13-23), and extensive chapter reviews at the ends of Chapters 6 and 12. 413 pp. $4.00.

H B

HARPER & BROTHERS • 49 East 33d Street, N.Y. 16

NEW Books from Prentice-Hall

CALCULUS WITH ANALYTIC GEOMETRY

by ANGUS E. TAYLOR, Univ. of California, Los Angeles

An efficient, modern treatment of calculus with analytic geometry, necessary for the training of scientists and engineers. Incorporates the best modern practice in treatment of the function concept and other aspects of fundamental theoretical issues. Presents ample material, helpful both to student and teacher: illustrative examples, diagrams, and interesting, well-graded sets of exercises ranging from simple ones designed for aid in grasping concepts and mastering techniques to complex exercises designed to challenge the student's insight, comprehension and ability to analyze, formulate, and resolve difficulties. Calculus is used wherever practicable to present topics in analytic geometry.

768 pp. *Published March 1959* *Text price $8.50*

INDUSTRIAL MATHEMATICS

by MATTHIAS THIES

In an elementary and comprehensive review, this new book provides a refresher course in those arithmetical operations without which no problem could be solved. (These indispensable operations include addition, subtraction, multiplication, division, powers, and roots.) *No previous knowledge of algebra, geometry or trigonometry* is required to understand the more advanced operations presented later in the book. Every topic covered has been developed and fully tested by the author and his associates in more than 100 Twelve-week Shop Math classes at vocational and technical schools during the past eight years.

Approx. 384 pp. *Published April 1959* *Text price $6.75*

HANDBOOK OF CALCULUS, DIFFERENCE AND DIFFERENTIAL EQUATIONS

by EDWARD J. COGAN, Sara Lawrence College and ROBERT Z. NORMAN, Dartmouth College

Especially useful as an accompaniment to any calculus text or in courses in Differential Equations or in Actuarial Math. The book discusses the concept of a function and operations on functions. It includes: Functions whose domains are integers and real numbers, laws for differentiation, list of derivatives of basic functions, differential equations, finite differences, properties of finite integrals, linear difference equations with constant coefficients. A half semester of calculus is a practical prerequisite.

288 pp. *Published 1958* *Text price $4.50*

To receive approval copies promptly, write: Box 903

PRENTICE-HALL, Inc.

Englewood Cliffs, New Jersey

TEXTBOOKS IN MATHEMATICS

CALCULUS

Richard E. Johnson and Fred L. Kiokemeister

Published this month. Emphasizing rigor and the intuitive meaning of the limit concept, this new text defines limits in the epsilon-delta manner and uses this definition to prove all limit theorems under discussion. Special features of the book include the early introduction of the definite integral (preceded by the applications of the derivative), and a separate chapter on vectors. Problems, answers, worked-out solutions, and tables (including a table of integrals) are strategically placed throughout the text to facilitate student comprehension of the theory and applications of the calculus.

ELEMENTS OF MATHEMATICS

J. Houston Banks

This freshman text stimulates student interest and curiosity with challenging problems and an original approach to traditional material. Book covers the elementary concepts of number, proof, measurement, and function while stressing the interdependence of all branches of mathematics. After postulating the natural numbers and their properties, the book develops the complex number system. Interwoven with this development is the notion of function, group, and field—through which algebra is developed along with the number system. Problems and questions are included in each chapter.

INTRODUCTORY MATHEMATICAL ANALYSIS

Edgar D. Eaves and Robert L. Wilson

Written especially for students who have had only one year of high school algebra and plane geometry, this textbook integrates the simple concepts of algebra, trigonometry, analytic geometry, and calculus into a unified course. Stressing the Rate Problem and the Rate Problem Reversed (using only polynomial functions) the book includes over 1,200 problems and examples to illustrate the meaning of new definitions and concepts. Answers to odd-numbered problems are included in the text.

For examination copies, write to Arthur B. Conant:

ALLYN AND BACON College Division

FROM ALLYN AND BACON

ARITHMETIC FOR COLLEGE STUDENTS

Edwin I. Stein

This book is a complete basic text in arithmetic designed for refresher courses for college students and also for courses in teaching elementary arithmetic. With its unique arrangement of thousands of graded examples in computation and the inclusion of numerous varied applications, it emphasizes both the mathematical and social aspects of arithmetic.

Elementary Algebra for College Students

Myron R. White

This book presents beginning algebra as a natural outgrowth of the student's experiences in arithmetic. The number system is extended to include general numbers, negative numbers, and irrational numbers. The formula and the equation are emphasized as the foundations of algebra. Over 6,000 graded exercises, divided into average and difficult groups, insure ample practice material for even exceptional students.

Intermediate Algebra for College Students

Myron R. White

The companion volume to *Elementary Algebra,* this book may also be used after any beginning algebra text. Outstanding features include clarity of presentation (both volumes are printed in two colors) and the careful consideration given to the order of topics. Logarithms and exponents appear early in the text; graphs are introduced where they are appropriate to the solution of a particular type of equation rather than in a single chapter. Thousands of graded exercises, plus cumulative reviews at the end of each chapter, are also included.

PLANE GEOMETRY FOR COLLEGE STUDENTS

William C. Stone

Book presents a traditional approach to plane geometry with various concepts of solid geometry included in several of the chapters. Each text-proved theorem starts with a plan of attack which encourages students to think independently. Book also includes a separate chapter which reviews the common forms of reasoning and gives training in the most useful laws of logic.

150 Tremont Street, Boston 11, Mass.

COLLEGE ALGEBRA AND TRIGONOMETRY
WILLIAM L. HART

Published this month, this modern presentation supplies the algebraic content beyond the third semester which is essential as preparation for a course in analytic geometry and calculus. The book provides a collegiate development of trigonometry, coordinated with its future role in calculus; defines the trigonometric functions in the classical fashion, but later makes the transition to the analytic trigonometric functions of numbers.

ANALYTIC GEOMETRY AND CALCULUS
WILLIAM L. HART

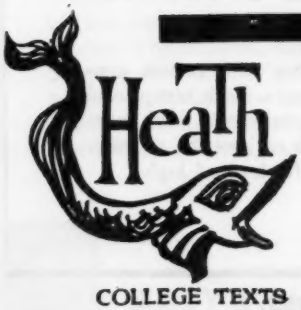
A highly successful modern introduction to the calculus, this text provides an early introduction to integration. No previous formal acquaintance with analytic geometry is assumed. The text achieves a high level of logical completeness throughout, yet is written with great clarity and simplicity. The book is well adapted to provide understanding of background theory as well as to develop mechanical proficiency. *648p. text* $7.00

COLLEGE ALGEBRA, 4th ed.
WILLIAM L. HART

The leading college algebra in the country, this text features a comprehensive collegiate review of intermediate algebra; all classical topics of college algebra; a novel introduction to signed numbers; a unique chapter on discrete probability. *420 p. text* $4.75

MATHEMATICAL ANALYSIS
E. J. CAMP

This volume presents an integrated treatment of topics from college algebra, trigonometry, analytic geometry, and calculus for the freshman year. The text is distinguished for its thorough coverage, clear explanations, careful illustrations, excellent motivation, and numerous stimulating exercises. *561p. text* $6.75



COLLEGE TEXTS

D. C. HEATH AND COMPANY

FROM MACMILLAN

THE THEORY OF GROUPS

by MARSHALL HALL, Jr., The Ohio State University

Divided into two parts, this text provides both the fundamentals of the theory of groups and a broad selection from the most recent and active areas of research in group theory. The first part of the text is designed for a course in the theory of groups; the second part may be used in a course or for reference purposes. *1959, 403 pages, $8.75*

ELEMENTARY MATRIX ALGEBRA

by FRANZ H. HOHN, University of Illinois

This new work presents in a clear and readable manner the topics of elementary matrix algebra that are most frequently used in the physical and social sciences. It includes a large collection of exercises plus a full treatment of the Laplace expansion and the determinantal properties of the adjoint matrix. *1958, 305 pages, $7.50* (text edition)

INTRODUCTION TO PROBABILITY AND STATISTICS

by B. W. LINDGREN and G. W. McELRATH, both, University of Minnesota

Primarily for statistics courses designed for engineering and industrial students, this text by a professor of mathematics and a professor of engineering contains numerous problems and illustrations plus such modern techniques as nonparametric tests. The problems apply theory to production and other industrial matters. *April 1959*

FUNDAMENTALS OF MATHEMATICS, Revised Edition

by MOSES RICHARDSON, Brooklyn College

Designed for the liberal arts student, this text emphasizes the fundamental concepts and applications of mathematics rather than its formal techniques. The author discusses new mathematical developments in the social and behavorial sciences. Some topics new to this edition: information theory, Boolean algebra in electrical networks, theory of games in strategy and mathematics of political structures. *1958, 507 pages, $6.50*

The Macmillan Company

60 FIFTH AVENUE, NEW YORK 11, N.Y.

PRINCIPLES OF MATHEMATICS

By C. B. ALLENDOERFER, University of Washington; and C. O. OAKLEY, Haverford College. 466 pages, $5.50

A new approach in both content and emphasis directs this important text toward the reform of the basic curriculum in mathematics. The emphasis is on an understanding of the methods of mathematical reasoning, the basic ideas of the subject, and a clear understanding of the reasons behind the mathematical processes. For students who have completed a course in intermediate algebra and who need preparation for a standard calculus course.

PRINCIPLES OF MECHANICS

By JOHN L. SYNGE, Dublin Institute for Advanced Studies; and BYRON A. GRIFFITH, Executive Vice President, KCS Data Control, Toronto, Canada. Third Edition. Ready in March.

A general revision of a highly regarded book which, because of its fundamental approach, is used by physicists, mathematicians, astronomers, and engineers. The book gives an orderly connected account of classical mechanics. The fundamentals of mechanics are carefully introduced, and basic principles emphasized. Major changes include: the addition of a new part intended primarily to meet the needs of students in theoretical physics, new numerical exercises and problems, answers to selected problems.

LOGIC IN ELEMENTARY MATHEMATICS

By ROBERT M. EXNER, Syracuse University; and MYRON S. ROSSKOPF, Columbia University. Ready in March.

Written for teachers and prospective teachers of mathematics courses, the book fills a need that has been apparent in mathematical literature for some time. The style and mode of development is simple, readable, interesting, and practical. At no point is logic developed for its own sake. Always the goal of application in mathematics is kept in mind.

PLANE TRIGONOMETRY

By GORDON P. FULLER, Texas Technological College. *Second Edition.* 282 pages, $4.75 (with Tables)

This greatly strengthened new edition emphasizes analytic trigonometry and brings it into proper balance with numerical trig. It includes clear, simple discussions and explanations; illustration of each new topic with problems worked in detail; practical problems taken from physics, surveying, and aviation; a complete and adequate treatment of logarithms; a discussion of trigonometric equations and inverse functions; a full chapter devoted to complex numbers. Price without Tables $4.25. Tables alone $1.50.

Send for copies on approval

McGraw-Hill Book Company, Inc.

330 West 42nd Street New York 36, N.Y.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.